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A Discrete Heisenberg Group which is not a Weakly Amenable

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Abstract

In this paper we prove that the Discrete Heisenberg group need not be weak amenable.

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1 Introduction

We provide approximation property of operator algebras associated with discrete groups. There are various notions of finite dimensional approximation properties for C^*- algebras and more generally operator algebras. Some of these (approximation properties) notations will be defined in this paper, the reader is referred to [2], [11], [12], [6] and [16]: Haagerup discovered that the reduced C^*- algebra $\mathbf{F}_n$ has the metric approximation property, Higson and Kasparov's resolution of the Baum-connes conjecture for the Haagerup groups. We study analytic techniques from operator theory that encapsulate geometric properties of a group. On approximation properties of group C^*- algebras is everywhere; it is powerful, important, backbone of countless breakthroughs. Weak amenability is strictly weaker than amenability and passes to closed subgroups. It is proved by De Canni'ere - Haagerup, Cowling and Cowling - Haagerup [5], [4] that real simple Lie groups of real rank one are weakly amenable (see also [15]), and by Haagerup [9] that real simple Lie groups of real rank at least two are not weakly amenable. For the latter fact, Haagerup proves that $SL(2,\mathbb{Z}) \rtimes \mathbb{Z}^2$ is not weakly amenable. (See also [10]) The notion of weak amenability for groups was introduced by Cowling and Haagerup [4]. Haagerup proved that all connected simple Lie groups with finite center and real rank greater than or equal to two are not weakly amenable [9]. Later, Dorofaeff proved that this result also holds for such Lie groups with infinite center [7]. A weaker approximation property defined in terms of completely bounded Fourier multipliers was introduced by the Haagerup and Kraus [8] Haagerup and Kraus have provided in [8] a detailed characterisation of AP.

This paper is organized as follows. In section 2 we recall some results about Approximation Property(AP), Weakly amenable and Herz- Schur multipliers.

Section 3 provides some detail of Discrete Heisenberg group and this group need not be weak amenability.

Our main result in this direction is the following.

Theorem 1.1. *The discrete Heisenberg Group need not be weak amenability.*

2 Preliminary

An important class of C^*- algebras arises in the study of groups. If we assume that the G is a discrete group then the functions δ_g form a basis for the Hilbert space $\ell^2(G)$ of square summable functions on G.

Definition 2.1. [6] The *left regular representation* $\lambda : \mathbb{C}[G] \rightarrow B(\ell^2(G))$ is defined by $\lambda(s)\delta_t(r) = \delta_t(s^{-1}r) = \delta_{st}(r)$ for $s, r \in G$. The *right regular representation* is given by $\rho(s)\delta_t(r) = \delta_t(rs) = \delta_{ts^{-1}}(r)$ for $s, r \in G$.

Definition 2.2. [6] The *reduced group C^*- algebra G*, denoted by $C^*_r(G)$ is the completion of $\mathbb{C}[G]$ in the norm given, for $c \in \mathbb{C}[G]$, by $\|c\|_\lambda = \|\lambda(c)\|$

We begin with some definition of Haagerup and Kraus [10]. We recall the Fourier algebra $A(G) := \{f : f(t) = \langle \lambda(t)\xi \ , \ \eta\rangle \text{ for some } \xi, \eta \in L_2(G)\}$ is the space of all coefficients function of the left regular representation λ. Given $f \in A(G)$, its norm is given by $\|f\| = \inf\{\|\xi\| \, \|\eta\| : \ f(t) = \langle\lambda(t)\xi \, , \, \eta\rangle\}$. With this norm, $A(G)$ is a Banach algebra with the pointwise multiplication [10]. A complex-valued function ϕ on G is a *multiplier* [1] for $A(G)$ if the linear map $m_\phi(f) = \phi f$ sends $A(G)$ to $A(G)$. A multiplier ϕ is called completely bounded [1] if the operator $m_\phi : L(G) \longrightarrow L(G)$ induced by m_ϕ is completely bounded. For a function ϕ on G and $C \geq 0$. We define the multiplier [10],[1] $m_\phi : \lambda(f) \longrightarrow \lambda(\phi f)$ is completely bounded on $C^*_\lambda(G)$ and $\|m_\phi\|_{cb} \leq C$.

Definition 2.3. [2] The discrete group G is *amenable* if and only if there is an approximate identity consisting of positive definite functions.

Definition 2.4. [1] An approximate identity on G is a sequence (ϕ_n) of finitely supported functions such that ϕ_n uniformly converge to the constant function 1. We say that discrete G is *weakly amenable* if there is an approximate identity (ϕ_n) such that $C := \sup \|M_{\phi_n}\|_{cb} < \infty$.

Definition 2.5. [8] The discrete group G has the *approximation property* (AP) if there is a net $\{\phi_\alpha\}$ in $A(G)$ such that $M_{\phi_\alpha} \longrightarrow id_{A(G)}$ in the stable point-norm topology on $A(G)$.

We have the following important result by Haagerup and Kraus [10].

Theorem 2.6. *G is a locally compact group and Γ is a lattice in G, then G has the AP if and only if G has the AP.*

The AP has some nice stability properties that weak amenability does not have, e.g.,

Theorem 2.7. *[10] If H is a closed normal subgroup of a locally compact group G such that both H and G/H have the AP, then G has the AP.*

This implies that the group $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ has the AP, but it was proven in [8] that this group is not weakly amenable, so the AP is strictly weaker than weak amenability. A natural question to ask is which groups do have the AP. When this property was introduced, it was not clear that there even exist groups without it, but it was conjectured by the Haagerup and Kraus that $SL(3,\mathbb{Z})$ would be such a group. This conjecture was recently proved by Lafforgue and de la Salle [14].

We say that the uniform Roe algebra [13], $C_u^*(G)$, is the C^*- algebra completion of the algebra of bounded operators on $\ell^2(X)$ which have finite propagation. According to Roe [16] G has the invariant approximation property (IAP) if

$$C_\lambda^*(G) = C_u^*(G)^G.$$

Next, we define the set of fixed points of $C_u^*(G,S)^G$:

Definition 2.8. We define

$$C_u^*(G,S)^G = \{T \in C_u^*(G,S)\,;\, Ad(\rho_t \otimes id)T = T \text{ for all } t \in G\}.$$

We now define Joachim Zacharias's IAP with coefficients (SIAP):

Definition 2.9. We say that a discrete group G has the *strong invariant translation approximation property* (SIAP) if for any closed subspace S of the compact operators $\mathcal{K}$ (on $\ell^2(\mathbb{N})$). We have an isomorphism

$$C_u^*(G,S)^G = C_\lambda^*(G) \otimes S \text{ holds.}$$

Author shows that the stability properties of the strong invariant approximation property [12]and Strong invariant approximation property for discrete groups[11].

Proposition 2.10. *[8] The semi direct product of two discrete groups with the AP has the AP.*

Remark 2.11. For discrete groups we have the following implications:

$$\text{Amenability} \implies \text{weak amenability} \Longrightarrow \text{AP} \Longrightarrow \text{exactness.}$$

The first implication is not an equivalence: the non-abelian free groups are weakly amenable, but they are not amenable. The second implication is proved by Haagerup and Kraus showed in [8] and also this implication is not an equivalence: a counter-example is given by $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$; this group has the AP [8]. But it was proved in [8] that it is not weakly amenable. The third implication is not an equivalence: Haagerup and Kraus showed in [8] that $SL(2,\mathbb{Z})$ is an exact group without AP.

3 The Discrete Heisenberg Group

The discrete *Heisenberg group* [6] $\mathbb{H}_3$ can be defined abstractly as the group generated by elements a and b such that the commutator $c = aba^{-1}b^{-1}$ is central. And also $\mathbb{H}_3$ is the multiplicative group of all matrices of the form

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \ : \ a, b, c \in \mathbb{Z} \right\}$$

This group is generated by

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } W = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Lemma 3.1. *The mapping*

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (c, b, a),$$

is isomorphism .

Proof. Let $a, b, c, x, y, z \in \mathbb{Z}$. We define a mapping by

$$\phi : \mathbb{H}_3 \longrightarrow \mathbb{R}^3$$

by

$$\phi \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = (c, b, a)$$

Let

$$A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{H}_3, \text{ and } B = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{H}_3,$$

Consider

$$\begin{aligned} \phi(A)\phi(B) &= (c, b, a)\,(z, y, x) \\ &= (cz, by, ax) \\ &= \phi(AB) \;\forall\; A,\, B \in \mathbb{H}_3 \end{aligned}$$

Therefore ϕ is homomorphism from $\mathbb{H}_3$ into $\mathbb{R}^3$. □

A more convenient way then matrices to denote elements in this group is by three-triples of numbers. Using this notations [6]

$$\mathbb{H}_3 = \{(x, y, z) \;:\; x, y, z \in \mathbb{Z}\},$$

and the multiplicative operation of elements can be written:

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy').$$

This notation makes computations in the Heisenberg group much more efficient. The following are computational facts about $\mathbb{H}_3$.

Proposition 3.2. *Let $a, b, c, x, y, z \in \mathbb{Z}$. Then the multiplication in $\mathbb{H}_3$ satisfies the following equations:*

1. $(x, y, z)^{-1} = (-x, -y, -z + xy)$;
2. $(x, y, z)\,(a, b, c)\,(x, y, z)^{-1} = (a, b, c + xb - ay)$;
3. $[(x, y, z)\,, (a, b, c)] = (0, 0, -ay + xb)$;
4. *In particular* $[(0, 1, 0)\,, (0, 0, 1)] = (1, 0, 0)$;
5. $[(1, 0, 0)\,, (0, 1, 0)] = (0, 0, 1)$;

Proof. Consider

$$(x, y, z)(-x, -y, -z + xy) = (0, 0, 0).$$

Thus

$$(x, y, z)^{-1} = (-x, -y, -z + xy).$$

Consider

$$\begin{aligned}(x, y, z)(a, b, c)(x, y, z)^{-1} &= (x + a, y + b, z + c + xb)(-x, -y, -z + xy) \\ &= (a, b, c + xb - ay).\end{aligned}$$

Consider

$$\begin{aligned}[(x, y, z), (a, b, c)] &= (x, y, z)(a, b, c)(x, y, z)^{-1}(a, b, c)^{-1} \\ &= (a, b, c + xb - ay)(-a, -b, -c + ab) \\ &= (0, 0, -ay + xb).\end{aligned}$$

In particular, $[(0, 1, 0), (0, 0, 1)] = (1, 0, 0)$ and $[(1, 0, 0), (0, 1, 0)] = (0, 0, 1)$. By using Lemma 3.1:

$$[U, V] = W.$$

□

Lemma 3.3. *With the usual notation:*

1. $W := UVU^{-1}V^{-1}$;
2. $WU = UW$;
3. $WV = VW$.

Proof. Equations (2) and (3) are obvious from equation (1). □

For example $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $AB = (1, 1, 1)$. It is important to note that $(x, y, z)^{-1} = (-x, -y, -z+xy)$, and the identity of $\mathbb{H}_3$, $I_3 = (0, 0, 0)$. Since elements of $\mathbb{H}_3$ can be written in three-tuples of numbers, (x, y, z), it is natural to think of its Cayley graph embedded in $\mathbb{Z}^3 \subseteq \mathbb{R}^3$.

By using Lemma 3.3, we have the following Proposition:

Proposition 3.4.

$$\mathbb{H}_3 = \left\{U, V, W \ : \ W := UVU^{-1}V^{-1},\ U^3 = 1,\ V^3 = 1, \textit{and}\ W^3 = 1\right\}$$

Proposition 3.5. *The center of* $\mathbb{H}_3$ *is isomorphic to the additive group* $\mathbb{Z}$.

Proof. By using Proposition 3.2(2), we have center $\mathbb{Z}(\mathbb{H}_3)$ coincides with $\mathbb{Z} \times 0 \times 0 \subseteq \mathbb{H}_3$ and from Proposition 3.2(3) $\mathbb{Z}(\mathbb{H}_3) = [\mathbb{H}_3, \mathbb{H}_3] = \mathbb{Z}$. □

Remark 3.6. Let $\mathbb{H}_3$ be a discrete group such that $\mathbb{Z}$ is abelian and $\mathbb{H}_3$ is a central extension of $\mathbb{Z}$ by $\mathbb{Z}^2$. That is to say, we have a short exact sequence

$$1 \longrightarrow \mathbb{Z} \stackrel{i}{\longrightarrow} \mathbb{H}_3 \stackrel{\pi}{\longrightarrow} \mathbb{Z}^2 \longrightarrow 1,$$

where $\mathbb{Z}^2$ is normal subgroup of $\mathbb{H}_3$ and $\mathbb{Z}(\mathbb{H}_3) = \mathbb{Z}$. We also assume that $\mathbb{H}_3$ admits a continuous section over $\mathbb{Z}$. We define $\mathbb{H}_3 = \mathbb{Z}^2 \times \mathbb{Z}$ as set. The group law on $\mathbb{H}_3$ is given by $(\alpha, a)(\beta, b) = (\alpha\beta\phi(a, b), a + b)$ where $\phi : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}^2$ is a 2− cocycle, i.e $\phi(a, b)\phi(a + b, c) = \phi(a, b + c)\phi(b, c)$. It is a simple computation to see that this determines a group law on $\mathbb{H}_3$.

1.

$$\begin{aligned} ((\alpha, a)(\beta, b))(\gamma, c) &= (\alpha\beta\phi(a, b), a + b)(\gamma, c) \\ &= (\alpha\beta\gamma\phi(a, b)\phi(a + b, c), a + b) . \end{aligned}$$

$$\begin{aligned} (\alpha, a)((\beta, b)(\gamma, c)) &= (\alpha, a)(\beta\gamma\phi(b, c), b + c)) \\ &= (\alpha\beta\gamma\phi(a, b)\phi(a + b, c), a + b) . \end{aligned}$$

Therefore

$$((\alpha, a)(\beta, b))(\gamma, c) = (\alpha, a)((\beta, b)(\gamma, c)) .$$

2. Given a, ϕ, the identity (β, y) of the group must satisfy

$$(\alpha, 0)(\beta, y) = (\alpha\beta\phi(0, y), y) = (\alpha, 0).$$

So $y = 0$ and $\beta\phi(0, y) = 1$.Then $(\phi(0, 0)^{-1}, 0)$ is the identity.

3. So if $(\alpha, x)^{-1} = (\beta, y)$, we have $(\alpha\beta\phi(x, y), x + y) = (\phi(0, 0)^{-1}, 0)$. Thus $y = -x$ and $\beta^{-1} = \alpha\phi(x, y)\phi(0, 0)$.

Remark 3.7. Now points a and b in $\mathbb{Z}^2$, and choose m and n in $\mathbb{H}_3$. Then $mnm^{-1}n^{-1}$ in $\mathbb{Z}$. Since $i(mnm^{-1}n^{-1}) = a + b - a - b = 0$. Let $m = (\alpha, a)$ and $n = (\beta, b)$. We have a map $\lambda : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}^2$ define by $\lambda(a, b) = mnm^{-1}n^{-1}$.

Proposition 3.8. *We have the following:*

1. $\lambda(a + b, c) = \lambda(a, c)\lambda(b, c)$,

2. $\lambda(a, b + c) = \lambda(a, b)\lambda(a, c)$,

3. $\lambda(a, a) = 1$,

4. $\lambda(a,b) = \lambda(b,a)^{-1}$,
5. $\lambda(a,b) = \frac{\phi(a,b)}{\phi(b,a)}$.

Proof. First we show the last part. Let $m = (1,a)$ and $n = (1,y)$, so

$$\begin{aligned} m^{-1}n^{-1} &= (nm)^{-1} \\ &= (\phi(b,a), a+b)^{-1} \\ &= \left\{\phi(b,a)^{-1}\phi(a+b,-a-b)^{-1}\phi(0,0)^{-1}, -a-b\right\}. \end{aligned}$$

Then

$$mnm^{-1}n^{-1} = \left\{\phi(a,b)\phi(b,a)^{-1}\phi(0,0)^{-1}, 0\right\}.$$

However, we have

$$\lambda(a,a) = 1 = \frac{\phi(a,a)}{\phi(a,a)}\phi(0,0)^{-1}$$

So the identity of the group is actually $(1,0)$ and

$$\lambda(a,b) = \frac{\phi(a,b)}{\phi(b,a)}.$$

Next we consider the identies (by using Remark 3.6)

$$\phi(a,b)\phi(a+b,c) = \phi(a,b+c)\phi(b,c),$$

$$\phi(b,c)\phi(a+c,b) = \phi(c,a+b)\phi(a,b),$$

and

$$\phi(a,c)\phi(a+c,b) = \phi(a,b+c)\phi(c,b).$$

It follows:

$$\begin{aligned} \lambda(a+b,c) &= \frac{\phi(a+b,c)}{\phi(c,a+b)} \\ &= \frac{\phi(a,b+c)\phi(b,c)}{\phi(c,a)\phi(a+c,b)} \\ &= \frac{\phi(a,c)\phi(b,c)}{\phi(c,a)\phi(c,b)} \\ &= \lambda(a,b)\lambda(b,c). \end{aligned}$$

Thus

$$\lambda(a+b,c)=\lambda(a,b)\lambda(b,c).$$

It follows:

$$\begin{aligned}\lambda(a,b+c) &= \frac{\phi(a,b+c)}{\phi(b+c,a)}\\ &= \frac{\phi(a,b)\phi(a+b,c)}{\phi(b,c+a)\phi(c,a)}\\ &= \frac{\phi(a,b)\phi(a,c)}{\phi(b,a)\phi(c,a)}\\ &= \lambda(a,b)\lambda(b,c).\end{aligned}$$

Thus

$$\lambda(a,b+c)=\lambda(a,b)\lambda(a,c).$$

□

Theorem 3.9. *A discrete Heisenberg Group need not be weak amenable.*

Proof. It follow that from Proposition 3.5 that the center $\mathbb{Z}(\mathbb{H}_3)=\mathbb{Z}\times 0\times 0=\mathbb{Z}$ and from Proposition 3.2 that $[\mathbb{H}_3,\mathbb{H}_3]=\mathbb{Z}(\mathbb{H}_3)=\mathbb{Z}$. The center $\mathbb{H}_3$ is an extension of $\mathbb{Z}$ by $\mathbb{Z}^2$. We have mentioned that $\mathbb{H}_3$ can be viewed as the semi direct product of $\mathbb{Z}^2$ by $\mathbb{Z}$. We have $\mathbb{H}_3=\mathbb{Z}^2\rtimes\mathbb{Z}$. We have

$$1\longrightarrow\mathbb{Z}\xrightarrow{i}\mathbb{H}_3\xrightarrow{\pi}\mathbb{Z}^2\longrightarrow 1,$$

Since $\mathbb{Z}$ and $\mathbb{Z}^2$ are finitely generated and they have AP and also weakly amenable. Since semi direct product of weakly amenable group have not weakly amenable group. Therefore $\mathbb{H}_3$ is not a weak amenability. □

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