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Research Article

Some Results of $\tau_1\tau_2$-δ Semiconnectedness and Compactness in Bitopological Spaces

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We are going to establish some results of $\tau_1\tau_2$-δ semiconnectedness and compactness in a bitopological space. Besides, we will investigate several results in $\tau_1\tau_2$-δ semiconnectedness for subsets in bitopological spaces. In particular, we will discuss the relationship related to semiconnectedness between the topological spaces and bitopological space. That is, if a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$-δ semiconnected, then the topological spaces (X, τ_1) and (X, τ_2) are δ-semiconnected. In addition, we introduce the result which states that a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$-δ semiconnected if and only if X and ϕ are the only subsets of X which are $\tau_1\tau_2$-δ semiclopen sets. Moreover, we have proved some results in compactness also. Altogether, several results of $\tau_1\tau_2$-δ semiconnectedness and compactness in a bitopological space have been discussed.

1. Introduction

The concept "bitopological space" was established by Kelly in [1]. He introduced this concept in his journal of London Mathematical Society in 1963. He initiated his study about bitopological space by using quasimetric and its conjugate. A quasimetric on a set X is a nonnegative real valued function $p(,)$ on the Cartesian product of $X(X \times X)$ that satisfies the following three axioms:

(1) $p(x, x) = 0$, $\forall x \in X$.

(2) $p(x, z) \leq p(x, y) + p(y, z)$, $\forall x, y, z \in X$.

(3) $p(x, y) = 0$ if and only if $x = y$, $\forall x, y \in X$.

However, the quasimetric cannot be a metric. Because the symmetric property does not hold for quasimetric. Moreover, every metric space is a topological space. But this is not true for bitopological space in general. Anyhow, bitopological spaces exist for quasimetric spaces. Maheshwari and Prasad [2] introduced semiopen sets in bitopological spaces in 1977. In 1987, the notion δ-open sets in bitopological spaces was introduced by Banerjee [3]. After that, Khedr [4] introduced and studied about $\tau_1\tau_2$- open sets. Later, Fukutake [5] defined one kind of semiopen sets in 1989. Recently, Edward Samuel and Balan [6] established the concept $\tau_1\tau_2$-δ semiopen sets in bitopological spaces. We have already published some properties of $\tau_1\tau_2$-δ semiopen/closed sets in bitopological spaces in [7]. Moreover, we have presented some results of $\tau_1\tau_2$-δ semiconnectedness in bitopological spaces in [8]. In this paper, we are going to discuss the following results:

(1) Let $\{A_\alpha : \alpha \in I\}$ be a family of $\tau_1\tau_2$-δ semiconnected subsets of a bitopological space (X, τ_1, τ_2) with $\bigcap A_\alpha \neq \phi$. Then $\bigcup_{\alpha\in I} A_\alpha$ is also $\tau_1\tau_2$-δ semiconnected.

(2) If a bitopological space $(X, \tau_{1s}, \tau_{2s})$ is $\tau_{1s}\tau_{2s}$-δ semiconnected, then (X, τ_1, τ_2) is $\tau_1\tau_2$-δ semiconnected.

(3) If a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$-δ semiconnected, then the topological spaces (X, τ_1) and (X, τ_2) are δ-semiconnected.

(4) A bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$-δ semiconnected if and only if X and ϕ are the only subsets of X which are $\tau_1\tau_2$-δ semiclopen sets.

(5) In a bitopological space (X, τ_1, τ_2), if X is $\tau_1\tau_2$-δ semiconnected then X cannot be expressed as the union of two sets $A(\neq \phi)$ and $B(\neq \phi)$ with $A \cap B = \phi$ such that A is τ_1-δ semiopen and B is τ_2-δ semiopen.

(6) In a bitopological space (X, τ_1, τ_2), if X cannot be expressed as the union of two nonempty sets A and B with $A \cap B = \phi$ such that A is τ_1-δ semiopen and B is τ_2-δ semiopen, then X does not contain any nonempty proper subset which is both τ_1-δ semiopen and τ_2-δ semiclosed.

(7) The union of any family of $\tau_1\tau_2$-δ semiconnected sets with a nonempty intersection is $\tau_1\tau_2$-δ semiconnected.

(8) Every $\tau_1\tau_2$-δ semicompact space is $\tau_1\tau_2$-δ compact.

(9) If Y is τ_1-δ semiclosed subset of a $\tau_1\tau_2$-δ semicompact space (X, τ_1, τ_2), then Y is τ_2-δ semicompact.

(10) If Y is τ_1-δ closed subset of a $\tau_1\tau_2$-δ semicompact space (X, τ_1, τ_2), then Y is τ_2-δ semicompact.

2. Materials and Methods

Let τ_i-int(A), τ_i-cl(A), τ_i-δint(A), τ_i-δcl(A), and τ_i-δscl(A) be the interior, closure, δ-interior, δ-closure, and δ-semiclosure of A with respect to the topology τ_i, respectively, $i = 1, 2$. Let τ_j-δint(A) and τ_j-δcl(A) are the δ-interior and δ-closure of A with respect to the topology τ_j, respectively, $j = 1s, 2s$, where τ_{1s} and τ_{2s} are semiregularization of τ_1 and τ_2, respectively.

Definition 1. For a nonempty set X, we define two topologies τ_1 and τ_2 (τ_1 and τ_2 may be the same or distinct). Then the triple (X, τ_1, τ_2) is called a bitopological space.

Definition 2 (see [1]). Let A be subset of a bitopological space (X, τ_1, τ_2). Then A is called $\tau_1\tau_2$-open, if $A \in \tau_1 \cup \tau_2$. Complement of $\tau_1\tau_2$-open set is called $\tau_1\tau_2$-closed set.

Definition 3 (see [9]). Let A be subset of a bitopological space (X, τ_1, τ_2). Then A is called

(1) τ_{12}-regular open, if $A = \tau_1$-int$(\tau_2$-cl$(A))$;

(2) τ_{21}-regular open, if $A = \tau_2$-int$(\tau_1$-cl$(A))$;

(3) $\tau_1\tau_2$-semiopen, if $A \subseteq \tau_2$-cl$(\tau_1$-int$(A))$;

(4) $\tau_1\tau_2$-semiclosed, if $A \supseteq \tau_2$-int$(\tau_1$-cl$(A))$.

Definition 4 (see [9]). Let A be subset of bitopological space (X, τ_1, τ_2). Then,

(1) A is said to be τ_1-δ open set, if, for $x \in A$, there exists τ_{12}-regular open set G such that $x \in G \subset A$. Complement of τ_1-δ open set is called τ_1-δ closed set;

(2) A is said to be τ_2-δ open set, if for $x \in A$, there exists τ_{21}-regular open set G such that $x \in G \subset A$. Complement of τ_2-δ open set is called τ_2-δ closed set;

(3) Collection of all τ_1-δ open sets and τ_2-δ open sets are denoted by τ_{1s} and τ_{2s} respectively. And also $\tau_{1s} \subset \tau_1$ and $\tau_{2s} \subset \tau_2$.

Definition 5 (see [6]). Let A be subset of bitopological space (X, τ_1, τ_2). Then, A is called $\tau_1\tau_2$-δ semiopen set, if there exists an τ_1-δ open set U such that $U \subseteq A \subseteq \tau_2$-cl$(U)$.

Complement of τ_1-δ open set is called τ_1-δ closed set.

Definition 6 (see [6]). A subset Y is called a $\tau_1\tau_2$-δ semidisconnected subset of a bitopological space (X, τ_1, τ_2), if $\exists\, \tau_1\tau_2$-δ semiopen sets U, V such that $U \cap Y \neq \phi \neq V \cap Y, U \cap V \cap Y = \phi$ and $Y \subseteq U \cup V$. Otherwise Y is called a $\tau_1\tau_2$-δ semiconnected subset.

Definition 7 (see [6]). A bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-δ semiconnected space, if X cannot be expressed as the union of two disjoint sets $A(\neq \phi)$ and $B(\neq \phi)$ such that $(A \cap \tau_1$-δscl$(B)) \cup (B \cap \tau_2$-δscl$(A)) = \phi$. Suppose X can be so expressed, then X is called $\tau_1\tau_2$-δ semidisconnected space and we write $X = A \backslash B$ and it is called $\tau_1\tau_2$-δ semiseparation of X.

Definition 8. A nonempty collection $C = \{A_i : i \in I\}$ is called a $\tau_1\tau_2$-δ semiopen cover of a bitopological space (X, τ_1, τ_2), if $X = \bigcup_{i \in I} A_i$ and $C \subseteq \tau_1$-δSO$(X, \tau_1, \tau_2) \cup \tau_2$-$\delta$SO$(X, \tau_1, \tau_2)$ and C contains at least one member of τ_1-δSO(X, τ_1, τ_2) and one member of τ_2-δSO(X, τ_1, τ_2).

Definition 9. A cover $C = \{A_i : i \in I\}$ of a bitopological space is called $\tau_1\tau_2$-δ open cover of X, if $C \subseteq \tau_{1s} \cup \tau_{2s}$ and $C \cap \tau_{1s} \neq \phi$, $C \cap \tau_{2s} \neq \phi$ and $X = \bigcup C$.

Definition 10. A bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-δ compact, if every $\tau_1\tau_2$-δ open cover of X has a finite subcover.

Definition 11. A bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-δ semicompact, if every $\tau_1\tau_2$-δ semiopen cover of X has a finite subcover.

3. Results

3.1. Semiconnectedness

Proposition 12 (see [8]). *Let $\{A_\alpha : \alpha \in I\}$ be family of $\tau_1\tau_2$-δ semiconnected subsets of a bitopological space (X, τ_1, τ_2) such that $\bigcap_{\alpha \in I} A_\alpha \neq \phi$; then $\bigcup_{\alpha \in I} A_\alpha$ is also $\tau_1\tau_2$-δ semiconnected.*

Proof. Let $x_0 \in \bigcap_{\alpha \in I} A_\alpha$. Now, assume that $Y = \bigcup_{\alpha \in I} A_\alpha$ is not $\tau_1\tau_2$-δ semiconnected. Then there exist two $\tau_1\tau_2$-δ semiopen sets U and V if $U \cap Y \neq \phi \neq V \cap Y, U \cap V \cap Y = \phi$ and $Y \subseteq U \cup V$, then let $x_0 \in U$ (other case is similar). Now there exist $\alpha \in I$ such that $A_\alpha \cap V \neq \phi$ also $A_\alpha \cap U \neq \phi$ (since $x_0 \in A_\alpha$) and also $U \cap V \cap A_\alpha = \phi$ and $A_\alpha \subset U \cup V$ (since $A_\alpha \subset Y$), which shows that A_α is $\tau_1\tau_2$-δ semidisconnected subset and it is a contradiction. So, $\bigcup_{\alpha \in I} A_\alpha$ is $\tau_1\tau_2$-δ semiconnected. □

Proposition 13. *If a bitopological space $(X, \tau_{1s}, \tau_{2s})$ is $\tau_{1s}\tau_{2s}$-δ semiconnected, then (X, τ_1, τ_2) is $\tau_1\tau_2$-δ semiconnected.*

Proof. Suppose $(X, \tau_{1s}, \tau_{2s})$ is $\tau_{1s}\tau_{2s}$-δ semiconnected. Then X cannot be expressed as the union of two nonempty disjoint sets A and B such that $(A \cap \tau_{1s}$-δscl$(B)) \cup (B \cap \tau_{2s}$-$\delta$scl$(A)) = \phi$. Also A is τ_{1s}-δ semiopen and B is τ_{2s}-δ semiopen. Since $\tau_{1s} \subseteq \tau_1$ and $\tau_{2s} \subseteq \tau_2$, we have every τ_{1s}-δ semiopen and τ_{2s}-δ semiopen are τ_1-δ semiopen and τ_2-δ semiopen, respectively. Therefore, X cannot be expressed as the union of two nonempty disjoint sets A and B such that A is

τ_1-δ semiopen and B is τ_2-δ semiopen, respectively. Hence, (X,τ_1,τ_2) is $\tau_1\tau_2$-δ semiconnected. □

Proposition 14 (see [8]). *If a bitopological space (X,τ_1,τ_2) is $\tau_1\tau_2$-δ semiconnected, then the topological spaces (X,τ_1) and (X,τ_2) are δ-semiconnected.*

Proof. Since every τ_1-δ open set and τ_2-δ open set are τ_1-δ semiopen set and τ_2-δ semiopen set, respectively, So if (X,τ_1) and (X,τ_2) are δ-semidisconnected spaces then the bitopological space (X,τ_1,τ_2) becomes δ-semidisconnected. But this is impossible. So (X,τ_1) and (X,τ_2) are δ-semiconnected spaces. □

Proposition 15 (see [8]). *A bitopological space (X,τ_1,τ_2) is $\tau_1\tau_2$-δ semiconnected if and only if X and ϕ are the only subsets of X which are $\tau_1\tau_2$-δ semiclopen sets (simultaneously semiopen and semiclosed.)*

Proof. Consider a $\tau_1\tau_2$-δ semiconnected space (X,τ_1,τ_2); let $\phi \neq A \neq X$ and A is $\tau_1\tau_2$-δ semiclopen set; then $X = A \cup (X\backslash A)$ is $\tau_1\tau_2$-δ semidisconnected in the bitopological space, which is contradiction. So X and ϕ are the only subsets of X which are both $\tau_1\tau_2$-δ semiclopen sets. Conversely, let X and ϕ be the only subsets of X which are both $\tau_1\tau_2$-δ semiclopen sets. If the bitopological space is $\tau_1\tau_2$-δ semidisconnected, so there exists a $\tau_1\tau_2$-δ semidisconnection $X = A \cup B$ of the bitopological space. So $A = X\backslash B$ and $B = X\backslash A$; then A and B both are $\tau_1\tau_2$-δ semiclopen sets and each of them is neither X nor ϕ. This is a contradiction. Therefore, (X,τ_1,τ_2) is $\tau_1\tau_2$-δ semiconnected. □

Proposition 16. *If X is $\tau_1\tau_2$-δ semiconnected then X cannot be expressed as the union of two nonempty sets A, B with $A\cap B = \phi$ such that A is τ_1-δ semiopen and B is τ_2-δ semiopen.*

Proof. Assume that X can be expressed as the union of two nonempty disjoint sets A and B such that A is τ_1-δ semiopen and B is τ_2-δ semiopen, respectively. Since $A\cap B = \phi$, we have $A \subseteq B^c$. Then τ_2-$\delta\text{scl}(A) \subseteq \tau_2$-$\delta\text{scl}(B^c) = B^c$. Therefore, τ_2-$\delta\text{scl}(A) \cap B = \phi$. Similarly, we can prove $A \cap \tau_1$-$\delta\text{scl}(B) = \phi$. Hence, $(A\cap\tau_1$-$\delta\text{scl}(B))\cup(\tau_2$-$\delta\text{scl}(A)\cap B) = \phi$. This contradicts our supposition. Thus, X cannot be expressed as the union of two nonempty disjoint sets A and B such that A is τ_1-δ semiopen and B is τ_2-δ semiopen. □

Proposition 17. *If X cannot be expressed as the union of two disjoint sets $A(\neq \phi)$ and $B(\neq \phi)$ such that A is τ_1-δ semiopen and B is τ_2-δ semiopen, then X does not contain any nonempty proper subset which is both τ_1-δ semiopen and τ_2-δ semiclosed.*

Proof. Let X cannot be expressed as the union of two nonempty sets A, B with $A \cap B = \phi$ such that A is τ_1-δ semiopen and B is τ_2-δ semiopen. If X contains a nonempty proper subset A which is both τ_1-δ semiopen and τ_2-δ semiclosed. Then $X = A\cup A^c$, where A is τ_1-δ semiopen, A^c is τ_2-δ semiopen, and A, A^c are disjoint. This is a contradiction. Thus, X does not contain any nonempty proper subset which is both τ_1-δ semiopen and τ_2-δ semiclosed. □

Proposition 18. *The union of any family of $\tau_1\tau_2$-δ semiconnected sets with a nonempty intersection is $\tau_1\tau_2$-δ semiconnected.*

Proof. Take $A = \bigcup_{i\in I} A_i$, where each A_i is $\tau_1\tau_2$-δ semiconnected with $\bigcap A_i \neq \phi$. Suppose that A is not $\tau_1\tau_2$-δ semiconnected. Then $A = C \cup D$, where C and D are two nonempty disjoint sets such that $(C \cap \tau_1$-$\delta\text{scl}(D)) \cup (\tau_2$-$\delta\text{scl}(C) \cap D) = \phi$. Since A_i is $\tau_1\tau_2$-δ semiconnected and $A_i \subseteq A$, we have $A_i \subseteq C$ or $A_i \subseteq D$. Therefore, $\bigcup A_i \subseteq C$ or $\bigcup A_i \subseteq D$ as $A = \bigcup A_i = C \cup D$. Hence, $A \subseteq C$ or $A \subseteq D$. Since $\bigcap A_i \neq \phi$, we have $x \in \bigcap A_i$. Therefore, $x \in A_i$, $\forall i$. So, $x \in A$. Therefore, $x \in C$ or $x \in D$. Suppose $x \in C$. Since $C \cap D = \phi$, we have $x \notin D$. Therefore, $A \nsubseteq D$. Thus, $A \subseteq C$. This contradicts $A = C \cup D$. Thus, A is $\tau_1\tau_2$-δ semiconnected. □

3.2. Semicompactness

Proposition 19. *Every $\tau_1\tau_2$-δ semicompact space is $\tau_1\tau_2$-δ compact.*

Proof. Take (X,τ_1,τ_2) to be $\tau_1\tau_2$-δ semicompact. Let $C = \{A_i : i \in I\}$ be a pairwise open cover of X. Then $X = \bigcup_{i\in I} A_i$ and $C \subseteq \tau_{1s} \cup \tau_{2s}$ and C contains at least one member of τ_{1s} and one member of τ_{2s}. Since every τ_1-δ open set is τ_1-δ semiopen, we have $X = \bigcup_{i\in I} A_i$ and $C \subseteq \tau_1$-$\delta\text{SO}(X,\tau_1,\tau_2) \cup \tau_2$-$\delta\text{SO}(X,\tau_1,\tau_2)$ and C contains at least one member of τ_1-$\delta\text{SO}(X,\tau_1,\tau_2)$ and one member of τ_2-$\delta\text{SO}(X,\tau_1,\tau_2)$. Therefore, C is $\tau_1\tau_2$-δ semiopen cover of X. Since X is $\tau_1\tau_2$-δ semicompact, C has a finite subcover. Thus, (X,τ_1,τ_2) is $\tau_1\tau_2$-δ compact. □

Proposition 20. *If Y is τ_1-δ semiclosed subset of a $\tau_1\tau_2$-δ semicompact space (X,τ_1,τ_2) then Y is τ_2-δ semicompact.*

Proof. Let (X,τ_1,τ_2) be a $\tau_1\tau_2$-δ semicompact space. Let $\{A_\alpha : \alpha \in I\}$ be τ_2-δ semiopen cover of Y. Since Y is τ_1-δ semiclosed, Y^c is τ_1-δ semiopen. And $Y^c \cup \{A_\alpha : \alpha \in I\}$ be a $\tau_1\tau_2$-δ semiopen cover of X. Since X is $\tau_1\tau_2$-δ semicompact, $X = Y^c \cup (\bigcup_{i=1}^n A_i)$. Hence, $Y = \bigcup_{i=1}^n A_i$. Thus, Y is τ_2-δ semicompact. □

Proposition 21. *If Y is τ_1-δ closed subset of a $\tau_1\tau_2$-δ semicompact space (X,τ_1,τ_2) then Y is τ_2-δ semicompact.*

Proof. Since every τ_1-δ closed set is τ_1-δ semiclosed, Y is τ_1-δ semiclosed. Then by applying Proposition 20, Y is τ_2-δ semicompact. □

4. Discussion

In this paper, we have used the result that every τ_1-δ closed set is τ_1-δ semiclosed. Moreover, if A and B are two $\tau_1\tau_2$-δ semicompact subsets of X, then $A \cup B$ is also $\tau_1\tau_2$-δ semicompact. Besides, every $\tau_1\tau_2$-δ semicompact space is $\tau_1\tau_2$-δ

compact. The concept semiconnectedness and compactness is used in various parts of Mathematics. Simultaneously, the bitopological spaces have several applications in analysis, general topology, and theory of ordered topological spaces.

5. Concluding Remarks

In this paper, some results of $\tau_1\tau_2$-δ semiconnectedness and compactness in bitopological spaces have been discussed. Simultaneously, we have some important results which are related to connectedness and compactness. So we are interested to check whether those results will work for bitopological space or not. If the results fail to hold for bitopology, we are going to illustrate by examples. Further, we want to find how uniform continuity will work in bitopological spaces.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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