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Ridge estimator in a mixed Poisson regression model

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ABSTRACT

The generalized linear model approach of the mixed Poisson regression models (MPRM) is suitable for over-dispersed count data. The maximum likelihood estimator (MLE) is adopted to estimate their regression coefficients. However, the variance of the MLE becomes high when the covariates are collinear. The Poisson-Modification of Quasi Lindley (PMQL) regression model is a recently introduced model as an alternative MPRM. The variance of the proposed MLE for the PMQL regression model is high in the presence of multicollinearity. This paper adopts the ridge regression method for the PMQL regression model to combat such an issue, and we use several notable methods to estimate its ridge parameter. A Monte Carlo simulation study was designed to evaluate the performance of the MLE and the different PMQL ridge regression estimators by using their scalar mean square (SMSE) values. Further, we analyzed a simulated data and a real-life applications to show the consistency of the simulation results. The simulation and applications results indicate that the PMQL ridge regression estimators dominate the MLE when multicollinearity exists.

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1. Introduction

The Poisson regression model assists in modeling count responses with appropriate covariates. However, it is not a choice when the conditional variance of the count responses exceeds the conditional mean. This phenomenon is explained as over-dispersion or inflation of variation (see Greenwood and Yule 1920; Cameron and Trivedi 2013). When over-dispersion occurs, the generalized linear model (GLM) approach of mixed Poisson regression models are a well-known solution to explain the extra-variation of the count responses. To introduce these mixed Poisson regression models, the researchers have used different types of lifetime distributions for the Poisson conditional mean. For example, the Poisson-gamma/Negative binomial (NB) regression model; the Poisson-Inverse Gaussian regression model; the Poisson-Weighted exponential regression model; the Poisson-Generalized Lindley regression model, and the Poisson-Quasi Lindley regression model introduced by Greenwood and Yule (1920), Shoukri et al. (2021), Zamani et al. (2014), Wongrin and Bodhisuwan (2017), and Altun (2019), respectively.

Tharshan and Wijekoon (2021) also introduced a new continuous distribution named the Modification of the Quasi Lindley (MQL) distribution bounded to $(0, \infty)$. Its density function is given as

$$f_Y(y;\theta,\alpha,\delta) = \frac{\theta e^{-\theta y}}{(\alpha^3+1)\Gamma(\delta)}\left(\Gamma(\delta)\alpha^3 + (\theta y)^{\delta-1}\right);\ \ y>0, \theta>0, \alpha^3>-1, \delta>0, \qquad (1)$$

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where α and δ are shape parameters, θ is a scale parameter, and y is the respective random variable. Equation (1) presents the mixture of two non-identical distributions, exponential (θ), and gamma (δ, θ) with the mixing proportion, $p = \frac{\alpha^3}{\alpha^3+1}$. Since it has various flexible structural properties to model the Poisson conditional mean, i.e., its density function can be left-skewed, symmetrical, and right-skewed shapes with various rages of right-tail weights and dispersions, by amalgamating the Poisson distribution with the MQL distribution, Tharshan and Wijekoon (2022a) obtained an univariate mixed Poisson distribution named the Poisson-Modification of the Quasi Lindley (PMQL) distribution for the over-dispersed count data. Its probability mass function (pmf) is an explicit form and computationally flexible. Then, by using a re-parametrization technique, Tharshan and Wijekoon (2022b) derived its regression model to predict the over-dispersed count responses with a set of linear independent covariates based on the GLM approach. Authors have shown that the PMQL regression model provides better performance than some of the existing competent mixed Poisson regression models.

The main goal of these types of regression models is to estimate the regression coefficients so that the predicted value of the response counts close to the observed values. Since such mixed Poisson regression models' score functions with respect to the regression coefficients are nonlinear in the regression coefficients, the common estimator to estimate the unknown regression coefficients is the maximum likelihood estimator (MLE) which can be obtained by applying the iterative weighted least square (IWLS) algorithm. However, the MLE is not a good choice to estimate the unknown regression coefficients when there are high linear dependencies among the covariates. Such a problem is commonly known as the multicollinearity problem. As a consequence of the multicollinearity among covariates, a low statistical significance may occur for individual regressors. To combat this problem, some classical biased estimators are used in the literature to estimate the regression coefficients in the ordinary linear regression (OR) models (e.g., Hoerl and Kennard 1970a, 1970b; Liu 1993, 2003). The ridge regression method is one such classical method which has been highlighted consistently to be an alternative to the maximum likelihood (ML) estimation method in the presence of multicollinearity (see Farebrother 1976; Schaefer, Roi, and Wolfe 1984; Nomura 1988; Kibria 2003; Khalaf and Shukur 2005; Muniz and Kibria 2009; Månsson and Shukur 2011; Månsson 2012; Kibria and Lukman 2020). The ridge regression estimator (RE) provides a smaller mean square error (MSE) than the MLE when multicollinearity exists. In the OR model, ridge regression estimator is defined as

$$\hat{\beta}_{RE_{(OR)}} = (X'X + kI)^{-1}X'y, \tag{2}$$

where X is the data matrix of order $n \times (p+1)$ with p covariates, y is a $n \times 1$ vector of the response variable, $\hat{\beta}_{RE_{(OR)}}$ is the estimated vector of regression coefficients of order $(p+1) \times 1$ with intercept, I is the identity matrix of order $(p+1) \times (p+1)$, and $k \geq 0$ is the ridge parameter. The scalar mean square error (SMSE) of the ridge estimator $\hat{\beta}_{RE(OR)}$ is given as follows

$$\text{SMSE}(\hat{\beta}_{RE(OR)}) = \sigma^2 \sum_{j=1}^{p+1} \frac{s_j}{(s_j+k)^2} + k^2 \sum_{j=1}^{p+1} \frac{u_j^2}{(s_j+k)^2}, \tag{3}$$

where $\sigma^2 = \sum_{i=1}^{n} \frac{(y_i-\mu_i)^2}{n-p-1}$, $s_j(\ j = 1, 2, ..., p+1)$ is the jth eigenvalue of the matrix $X'X$, $u_j(j = 1, 2, ..., p+1)$ is the jth element of $U'\beta$, and U is the orthogonal matrix whose columns are the normalized eigenvectors of the matrix $X'X$. To estimate the ridge parameter, k, several estimation methods have been proposed. Some notable research works are: Hoerl and Kennard (1970a, 1970b), Nomura (1988), Kibria (2003), Khalaf and Shukur (2005), and Muniz and Kibria (2009). Further, the ridge regression method was extended to GLM by Segerstedt (1992).

When we focus on the ridge regression estimators for the count regression, there are very few works that have been done in the literature. The ridge regression method has been adopted in

Poisson regression, and negative binomial regression models by Månsson and Shukur (2011), and Månsson (2012), respectively. In their respective papers, the authors have adhered to some different ridge parameter estimators for the Poisson ridge regression (PRR) and the NB ridge regression (NBRR) estimators. They showed that regardless of which ridge parameter estimator is used for PRR and NBRR estimators, PRR and NBRR estimators give a lower scalar mean square error (SMSE) than the MLE in the presence of multicollinearity.

This paper aims to adopt the ridge regression method in the PMQL regression model to solve the multicollinearity problem. Further, we adhere to some notable ridge parameter estimators that are applicable for the PMQL ridge regression estimator ($RE_{(PMQL)}$). Then, the performance of MLE and $RE_{(PMQL)}$ based on different ridge parameter estimators will be compared in terms of the SMSE criterion by using an extensive Monte Carlo simulation study. In this simulation study, factors such as the degrees of correlation among the covariates (ρ), the sample size (n), the intercept (β_0), the number of covariates (p), and the over-dispersion parameters of the PMQL regression model (α, δ) are varied.

This paper is structured as follows: We present the PMQL distribution and its regression model in Sec. 2. $RE_{(PMQL)}$, MSE properties of $RE_{(PMQL)}$, and the ridge parameter estimators for $RE_{(PMQL)}$ are discussed in Sec. 3. Section 4 designs the Monte Carlo simulation study and discusses the results. A simulated data and a real-world applications are presented in Sec. 5. Finally, Sec. 6 presents the conclusion of the paper.

2. The PMQL distribution and its regression model

In this section, we present the PMQL distribution and its regression model.

2.1. The PMQL distribution

The PMQL distribution (Tharshan and Wijekoon 2022a) was obtained by letting the Poisson parameter follows the MQL distribution defined in Eq. (1). The probability mass function of the PMQL distribution is given as

$$f_Y(y) = \frac{\theta}{y!(\alpha^3+1)(1+\theta)^{y+\delta}\Gamma(\delta)}\left(\Gamma(\delta)\Gamma(y+1)\alpha^3(1+\theta)^{\delta-1} + \theta^{\delta-1}\Gamma(y+\delta)\right); \quad y = 0,1,2,\ldots,\theta > 0, \delta > 0, \alpha^3 > -1. \tag{4}$$

where y is the respective random variable and represents the total counts of an experiment. Its mean and variance are given,

$$E(Y) = \frac{\alpha^3+\delta}{(\alpha^3+1)\theta} = \mu, \tag{5}$$

and

$$Var(Y) = \mu + \mu^2\left(\frac{\alpha^3(\alpha^3+2+\delta(\delta-1))+\delta}{(\alpha^3+\delta)^2}\right), \tag{6}$$

respectively. Equation (4) represents a two-component mixture of geometric $\left(\frac{\theta}{1+\theta}\right)$ and negative binomial $\left(\delta, \frac{1}{1+\theta}\right)$ with the mixing proportion $p = \frac{\alpha^3}{\alpha^3+1}$. Further, it possesses to be unimodal and bimodal, and over-dispersed. The authors have shown that the PMQL distribution has various flexible structural properties for an over-dispersed count data, i.e., it has the potential to accommodate various horizontal symmetry, right-tail behaviors, and index of dispersion (see Tharshan and Wijekoon 2022a).

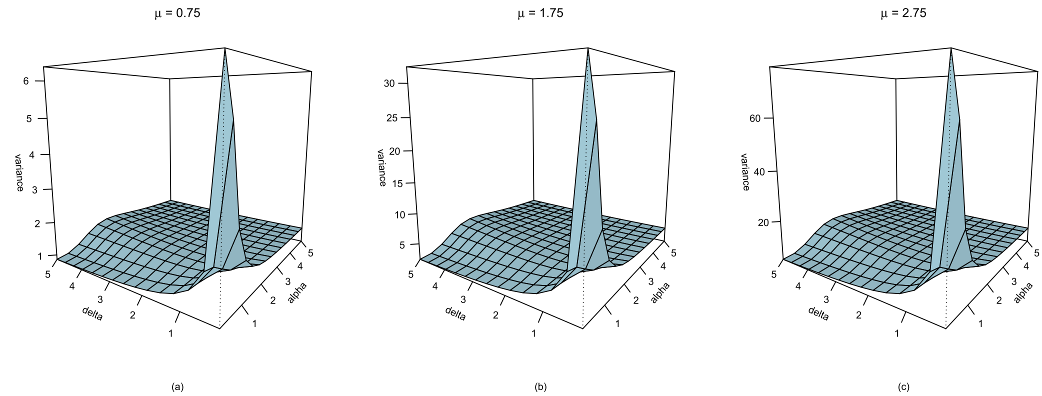

Figure 1. The variance function of PMQL regression model at $\mu = 1.50$ for different values of α, δ.

2.2. The PMQL regression model

The PMQL regression model (Tharshan and Wijekoon 2022b) is more useful than the Poisson and NB regression models since it is a flexible model to cover the various ranges of horizontal symmetry, right-tail heaviness, and index of dispersion of the response variable, y with covariates X. Let $y_1, y_2, ..., y_n$ be the random sample of n observations from the PMQL distribution. The link between p-dimensional covariates and the mean responses, y was taken as

$$\eta_i = g(\mu_i) = \log(\mu_i) = \sum_{j=0}^{p} \beta_j x_{ij} = x_i'\beta,\ i = 1, 2, ..., n, \tag{7}$$

where $x_i' = (1, x_{i1}, x_{i2}, ..., x_{ip})$ is the vector of covariates, and $\beta' = (\beta_0, \beta_1, ..., \beta_p)$ is a vector of unknown regression coefficients of order $(p+1) \times 1$, and α and δ are over-dispersion parameters. To approach the GLM, the PMQL distribution was re-parametrized based on the relationship between μ and θ given in Eq. (5) for a given set of α and δ values and the link between μ and p-dimensional covariates given in Eq. (7), i.e., by substituting $\theta_i = \frac{\alpha^3+\delta}{(\alpha^3+1)\exp(x_i'\beta)}, i = 1, 2, ..., n$ in Eq. (4), the pmf of the y_i for a given set of covariates x_i' was obtained as

$$f(y_i|x_i') = \frac{((\alpha^3+1)\exp(x_i'\beta))^{y_i}(\alpha^3+\delta)(\Gamma(\delta)\Gamma(y_i+1)\alpha^3 A_i^{\delta-1} + (\alpha^3+\delta)^{\delta-1}\Gamma(y_i+\delta))}{y_i!(\alpha^3+1)A_i^{y_i+\delta}\Gamma(\delta)}, \tag{8}$$

where $A_i = ((\alpha^3+1)\exp(x_i'\beta) + (\alpha^3+\delta)), i = 1, 2, ..., n$. The conditional mean and variance of the regression model are given,

$$E(Y_i|x_i') = \exp(x_i'\beta), \tag{9}$$

and

$$Var(Y_i|x_i') = \exp(x_i'\beta) + (\exp(x_i'\beta))^2\left(\frac{\alpha^3(\alpha^3+2+\delta(\delta-1))+\delta}{(\alpha^3+\delta)^2}\right), \tag{10}$$

respectively. Figure 1 depicts the surface plots of the variance function for the PMQL regression model at different values of μ, α, δ. According to this figure, for a given value of μ, the variance as a function of α or δ is not a monotonic function (there are several ups and downs) and it is high for small values of α, and δ. Further, for a given values of α and δ, the variance is increasing with μ.

The estimation of the unknown regression coefficients is commonly estimated by maximizing the following log-likelihood function of its pmf given in Eq. (8)

$$\begin{aligned}\ell(\beta,\alpha,\delta|y,x) &= \sum_{i=1}^{n} y_i \log\left((\alpha^3+1)\exp(x_i'\beta)\right) + n\log(\alpha^3+\delta) \\ &\quad - \sum_{i=1}^{n}\log(y_i!) - n\log(\alpha^3+1) - n\log(\Gamma(\delta)) \\ &\quad + \sum_{i=1}^{n}(\Gamma(\delta)\Gamma(y_i+1)\alpha^3 A_i^{\delta-1} + (\alpha^3+\delta)^{\delta-1}\Gamma(y_i+\delta)) - \sum_{i=1}^{n}(y_i+\delta)\log(A_i).\end{aligned} \tag{11}$$

The score function of the vector of regression coefficients, β is given as

$$\begin{aligned}S(\beta) = \frac{\partial\ell(\beta,\alpha,\delta|y,x)}{\partial\beta} &= \sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n}\frac{(y_i+\delta)(\alpha^3+1)\exp(x_i'\beta)x_i}{A_i} \\ &\quad + \sum_{i=1}^{n}\frac{\Gamma(\delta)\Gamma(y_i+1)\alpha^3(\delta-1)A_i^{\delta-2}(\alpha^3+1)\exp(x_i'\beta)x_i}{\Gamma(\delta)\Gamma(y_i+1)\alpha^3 A_i^{\delta-1} + (\alpha^3+\delta)^{\delta-1}\Gamma(y_i+\delta)}.\end{aligned} \tag{12}$$

Since Eq. (12) is non-linear in β, one can use the iteratively weighted least square (IWLS) algorithm (Fisher scoring method) (Dutang 2017) to obtain the maximum likelihood (ML) estimates. Let $\beta^{(s-1)}$ is the estimated value of β by the ML method with $(s-1)$ iterations. Then, the Fisher scoring method can be written as

$$\beta^{(s)} = \beta^{(s-1)} + I^{-1}(\beta^{(s-1)})S(\beta^{(s-1)}), \tag{13}$$

where $I(\beta^{(s-1)})$ be a $(p+1)\times(p+1)$ Fisher information matrix and the $S(\beta^{(s-1)})$ be the score function of the regression coefficients calculated at $\beta^{(s-1)}$. In the final step of the IWLS algorithm, the $\hat{\beta}_{MLE_{(PMQL)}}$ is obtained as

$$\hat{\beta}_{MLE_{(PMQL)}} = (X'\hat{W}X)^{-1}X'\hat{W}\hat{z}, \tag{14}$$

where $\hat{W} = diag\left(\frac{1}{(g'(\hat{\mu}_i))^2 Var(\hat{\mu}_i)}\right) = diag\left(\frac{\hat{\mu}_i(\alpha^3+\delta)^2}{(\alpha^3+\delta)^2+\hat{\mu}_i(\alpha^3(\alpha^3+2+\delta(\delta-1))+\delta)}\right)$, and $\hat{z}$ be a vector and its ith element is given as $g(\hat{\mu}_i) + (y_i-\hat{\mu}_i)g'(\hat{\mu}_i) = \log(\hat{\mu}_i) + \frac{(y_i-\hat{\mu}_i)}{\hat{\mu}_i}$.

The asymptotic covariance matrix of this estimator is given as

$$Cov\left(\hat{\beta}_{MLE_{(PMQL)}}\right) = (X'\hat{W}X)^{-1}, \tag{15}$$

and the asymptotic MSE and SMSE of this estimator are given as,

$$\begin{aligned}MSE(\hat{\beta}_{MLE_{(PMQL)}}) &= E\left((\hat{\beta}_{MLE_{(PMQL)}}-\beta)(\hat{\beta}_{MLE_{(PMQL)}}-\beta)'\right) \\ &= Cov(\hat{\beta}_{MLE_{(PMQL)}}) + (E(\hat{\beta}_{MLE_{(PMQL)}})-\beta)(E(\hat{\beta}_{MLE_{(PMQL)}})-\beta)' \\ &= (X'\hat{W}X)^{-1},\end{aligned} \tag{16}$$

and

$$\begin{aligned}SMSE(\hat{\beta}_{MLE_{(PMQL)}}) &= trace\left(MSE(\hat{\beta}_{MLE_{(PMQL)}})\right) \\ &= trace((X'\hat{W}X)^{-1}) = \sum_{j=1}^{p+1}\frac{1}{\lambda_j},\end{aligned} \tag{17}$$

respectively, where λ_j is the jth eigenvalue of the matrix $X'\hat{W}X$. When higher degrees of linear dependency among the covariates exist, the matrix $X'\hat{W}X$ is ill-conditioned and this matrix will have some small eigenvalues. Then, the SMSE($\hat{\beta}_{MLE_{(PMQL)}}$) given in Eq. (17) will be inflated, and then, we will have an erroneous interpretation of the relationship between the response and the covariates.

3. The PMQL ridge regression estimator

Now, we define a ridge regression estimator in the PMQL regression model to combat the multicollinearity problem as

$$\hat{\beta}_{RE_{(PMQL)}} = (X'\hat{W}X + kI)^{-1}(X'\hat{W}X)\hat{\beta}_{MLE_{(PMQL)}} = V_k\hat{\beta}_{MLE_{(PMQL)}}, \tag{18}$$

where $k \geq 0$ is the ridge parameter, $V_k = (X'\hat{W}X + kI)^{-1}(X'\hat{W}X)$, and I is the $(p+1) \times (p+1)$ identity matrix.

Asymptotic properties of PMQL ridge regression estimator

$$E(\hat{\beta}_{RE_{(PMQL)}}) = V_k\beta, \tag{19}$$

$$Cov(\hat{\beta}_{RE_{(PMQL)}}) = V_k Cov(\hat{\beta}_{MLE_{(PMQL)}})V_k' = V_k(X'\hat{W}X + kI)^{-1}, \tag{20}$$

and then the asymptotic bias, MSE, are given as,

$$\text{Bias}(\hat{\beta}_{RE_{(PMQL)}}) = E(\hat{\beta}_{RE_{(PMQL)}}) - \beta = (V_k - I)\beta = -k(X'\hat{W}X + kI)^{-1}\beta, \tag{21}$$

and

$$\begin{aligned}\text{MSE}(\hat{\beta}_{RE_{(PMQL)}}) &= E\Big((\hat{\beta}_{RE_{(PMQL)}} - \beta)(\hat{\beta}_{RE_{(PMQL)}} - \beta)'\Big) \\ &= Cov(\hat{\beta}_{RE_{(PMQL)}}) + \text{Bias}(\hat{\beta}_{RE_{(PMQL)}})\text{Bias}'(\hat{\beta}_{RE_{(PMQL)}}) \\ &= V_k(X'\hat{W}X + kI)^{-1} + k^2(X'\hat{W}X + kI)^{-1}\beta\beta'(X'\hat{W}X + kI)^{-1},\end{aligned} \tag{22}$$

respectively. Now, let us define an orthogonal matrix Γ whose columns are the normalized eigenvectors of the matrix $X'\hat{W}X$, a vector $\alpha = \Gamma'\beta$, and a diagonal matrix $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_{p+1}) = \Gamma'X'\hat{W}X\Gamma$. Then the asymptotic SMSE is derived by using the spectral decomposition as

$$\begin{aligned}\text{SMSE}(\hat{\beta}_{RE_{(PMQL)}}) &= trace(\text{MSE}(\hat{\beta}_{RE_{(PMQL)}})) \\ &= trace(Cov(\hat{\beta}_{RE_{(PMQL)}})) + \text{Bias}'(\hat{\beta}_{RE_{(PMQL)}})\text{Bias}(\hat{\beta}_{RE_{(PMQL)}}) \\ &= trace(V_k(X'\hat{W}X + kI)^{-1}) + k^2\beta'(X'\hat{W}X + kI)^{-2}\beta \\ &= trace((X'\hat{W}X + kI)^{-2}(X'\hat{W}X)) + k^2\beta'(X'\hat{W}X + kI)^{-2}\beta \\ &= trace((\Gamma'X'\hat{W}X\Gamma + kI)^{-2}\Gamma'X'\hat{W}X\Gamma) + k^2\beta'\Gamma(\Gamma'X'\hat{W}X\Gamma + kI)^{-2}\Gamma'\beta \\ &= trace((\Lambda + kI)^{-2}\Lambda) + k^2\beta'\Gamma(\Lambda + kI)^{-2}\Gamma'\beta \\ &= trace((\Lambda + kI)^{-2}\Lambda) + k^2\alpha'(\Lambda + kI)^{-2}\alpha \\ &= \sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j + k)^2} + k^2\sum_{j=1}^{p+1}\frac{\alpha_j^2}{(\lambda_j + k)^2},\end{aligned} \tag{23}$$

where $\alpha_j(\, j = 1, 2, ..., p+1)$ is the jth element of $\Gamma'\beta$. Note that the first sum of Eq. (23), $\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j+k)^2}$ is the total variance of regression coefficient estimates, and the second sum of Eq. (23), $k^2\sum_{j=1}^{p+1}\frac{\alpha_j^2}{(\lambda_j+k)^2}$ is the square bias of the estimator. Now, let us define them as:

$$\tau_1 = \sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j + k)^2}, \quad \text{and} \quad \tau_2 = k^2\sum_{j=1}^{p+1}\frac{\alpha_j^2}{(\lambda_j + k)^2}.$$

3.1. MSE properties of the PMQL ridge regression estimator

It is clear that the τ_1 is not inflated since the denominator is modified with $\lambda_j + k$ instead of λ_j. At $k = 0$, the value of the τ_1 is $\sum_{j=1}^{p+1} \frac{1}{\lambda_j}$ and τ_2 is zero. Then, at $k = 0$, the $\hat{\beta}_{RE_{(PMQL)}}$ equals the $\hat{\beta}_{MLE_{(PMQL)}}$.

Note that, by using (17) and (23) we get

$$\Delta = \text{SMSE}(\hat{\beta}_{MLE_{(PMQL)}}) - \text{SMSE}(\hat{\beta}_{RE_{(PMQL)}}) = \sum_{j=1}^{p+1} \frac{1}{\lambda_j} - \left(\sum_{j=1}^{p+1} \frac{\lambda_j}{(\lambda_j + k)^2} + k^2 \sum_{j=1}^{p+1} \frac{\alpha_j^2}{(\lambda_j + k)^2} \right), \tag{24}$$

and the estimator $\hat{\beta}_{RE_{(PMQL)}}$ is said to be superior to $\hat{\beta}_{MLE_{(PMQL)}}$ under the SMSE criterion iff $\Delta > 0$. Therefore, finding a suitable $k > 0$ such that $\Delta > 0$ is an important concept here.

For the ordinary linear regression model, Hoerl and Kennard (1970a, 1970b) have shown that there exists a $k > 0$ such that ridge regression estimator has a lower SMSE than the ordinary least square estimator (OLSE). In the count regression, Månsson and Shukur (2011) have shown that there exists a $k > 0$ such that SMSE of the PRR estimator is lower than the SMSE of MLE, and this property for the NBRR model is also shown by Månsson (2012). Similarly, we show that this property holds for the PMQL ridge regression model.

Proposition 1. *The total variance of the regression coefficient estimates of $\hat{\beta}_{RE_{(PMQL)}}$, ($\tau_1$) and squared bias of $\hat{\beta}_{RE_{(PMQL)}}$, ($\tau_2$) are continuous monotonically decreasing and increasing functions of k, respectively.*

Proof. The first derivative of the τ_1 is

$$\frac{\partial \tau_1}{\partial k} = -2 \sum_{j=1}^{p+1} \frac{\lambda_j}{(\lambda_j + k)^3}, \quad \text{and} \tag{25}$$

$$\lim_{k \to 0^+} \frac{\partial \tau_1}{\partial k} = -2 \sum_{j=1}^{p+1} \frac{1}{\lambda_j^2}. \tag{26}$$

Since $\lambda_j > 0$ for all j, Eq. (25) is always negative for all $k > 0$ and the derivative of the τ_1 in the neighborhood of the zero given in Eq. (26) is also negative.

The first derivative of τ_2 is

$$\frac{\partial \tau_2}{\partial k} = 2k \sum_{j=1}^{p+1} \frac{\alpha_j^2}{(\lambda_j + k)^3}, \quad \text{and} \tag{27}$$

$$\lim_{k \to 0^+} \frac{\partial \tau_2}{\partial k} = 0. \tag{28}$$

Since $\lambda_j > 0$ and $\alpha_j^2 > 0$ for all j, Eq. (27) is always positive for all $k > 0$ and the derivative of the τ_2 in the neighborhood of the zero given in Eq. (28) is zero.

Then, it is shown that τ_1 and τ_2 are continuous monotonically decreasing and increasing functions of k, respectively.

Proposition 2. *The SMSE($\hat{\beta}_{RE_{(PMQL)}}$) is a continuous monotonically decreasing function of k when $0 < k < \frac{1}{\alpha_{max}^2}$, where α_{max}^2 is the maximum element of $\alpha_j^2, j = 1, 2, ..., p + 1$.*

Proof. The first derivative of Eq. (23) is

$$\frac{\partial \ \mathrm{SMSE}(\hat{\beta}_{RE_{(PMQL)}})}{\partial k} = -2\sum_{j=1}^{p+1}\frac{\lambda_j}{(\lambda_j+k)^3} + 2k\sum_{j=1}^{p+1}\frac{\alpha_j^2}{(\lambda_j+k)^3}, \quad \text{and} \tag{29}$$

$$\lim_{k\to 0^+}\frac{\partial\,\mathrm{SMSE}(\hat{\beta}_{RE_{(PMQL)}})}{\partial k} = -2\sum_{j=1}^{p+1}\frac{1}{\lambda_j^2}. \tag{30}$$

We can note that if the individual ridge parameter $k_j < \frac{1}{\alpha_j^2}, \quad \forall j = 1, 2, \ldots, p+1,$ Eq. (29) is negative. Further, the derivative of the $\mathrm{SMSE}(\hat{\beta}_{RE_{(PMQL)}})$ in the neighborhood of the zero given in Eq. (30) is also negative. Then, it is clear that when $0 < k < \frac{1}{\alpha_{max}^2}$, Eq. (29) is always negative. So, it is shown that the $\mathrm{SMSE}(\hat{\beta}_{RE_{(PMQL)}})$ is a continuous monotonically decreasing function of k when $0 < k < \frac{1}{\alpha_{max}^2}$.

Now, from proposition 1, we can conclude that there is a possibility to find a value $k>0$. Further, the results of $\Delta=0$ at $k=0$ and proposition 2 indicate that the $\Delta>0$ when $0<k<\frac{1}{\alpha_{max}^2}$. Then, it is shown that there exists a $k>0$ such that $\Delta>0$.

Further, the optimal value of the k can be obtained by setting Eq. (29) to zero and solving for k. It is obtained as

$$k_j = \frac{1}{\alpha_j^2}, j = 1, 2, \ldots, p+1. \tag{31}$$

Lemma 1. *Let M be a positive definite (pd) matrix and c be a vector of nonzero constants. Then $M - c\,c' > 0$ iff $c'M^{-1}c < 1$ (Farebrother 1976).*

The following proposition discusses the condition that RE is superior to the MLE in PMQL regression model.

Proposition 3. *Let $b_k = Bias(\hat{\beta}_{RE_{(PMQL)}}) = -k(X'\hat{W}X + kI)^{-1}\beta$. Then* $\mathrm{MSE}(\hat{\beta}_{MLE_{(PMQL)}}) - \mathrm{MSE}(\hat{\beta}_{RE_{(PMQL)}}) > 0$ *if $b_k'(\Lambda^{-1} - (\Lambda+kI)^{-1}\Lambda(\Lambda+kI))^{-1}b_k < 1$.*

Proof. The difference between MSE of MLE and RE is derived by using the spectral decomposition as

$$\begin{aligned}
&\mathrm{MSE}(\hat{\beta}_{MLE_{(PMQL)}}) - \mathrm{MSE}(\hat{\beta}_{RE_{(PMQL)}})\\
&= (X'\hat{W}X)^{-1} - (X'\hat{W}X+kI)^{-1}(X'\hat{W}X)(X'\hat{W}X+kI)^{-1} - b_kb_k'\\
&= \Gamma\Gamma'(X'\hat{W}X)^{-1}\Gamma\Gamma' - \Gamma\Gamma'(X'\hat{W}X+kI)^{-1}\Gamma\Gamma'(X'\hat{W}X)\Gamma\Gamma'(X'\hat{W}X+kI)^{-1}\Gamma\Gamma' - b_kb_k'\\
&= \Gamma\Lambda^{-1}\Gamma' - \Gamma(\Lambda+kI)^{-1}\Lambda(\Lambda+kI)^{-1}\Gamma' - b_kb_k'\\
&= \Gamma(\Lambda^{-1} - (\Lambda+kI)^{-1}\Lambda(\Lambda+kI)^{-1})\Gamma' - b_kb_k'\\
&= \Gamma diag\left(\frac{1}{\lambda_j} - \frac{\lambda_j}{(\lambda_j+k)^2}\right)_{j=1,2,\ldots,p+1}\Gamma' - b_kb_k'\\
&= \Gamma diag\left(\frac{k(2\lambda_j+k)}{\lambda_j(\lambda_j+k)^2}\right)_{j=1,2,\ldots,p+1}\Gamma' - b_kb_k'.
\end{aligned}$$

Since $k(2\lambda_j+k) > 0(\ j = 1, 2, \ldots, p+1)$, the diagonal matrix $\Lambda^{-1} - (\Lambda+kI)^{-1}\Lambda(\Lambda+kI)^{-1}$ is a pd matrix, and b_k is a vector of nonzero constants. Then, by Lemma 1, if $b_k'(\Lambda^{-1} - (\Lambda+kI)^{-1}\Lambda(\Lambda+kI))^{-1}b_k < 1$. It completes the proof.

3.2. Estimation of the parameter k

The methods used by Hoerl and Kennard (1970a, 1970b), Nomura (1988), Kibria (2003), Khalaf and Shukur (2005), and Muniz and Kibria (2009) are adopted to estimate the ridge parameter k_j given in Eq. (31) with a single value $\hat{k}$. We define $\hat{k}_1$, $\hat{k}_2$, $\hat{k}_3 - \hat{k}_4$, $\hat{k}_5$ and $\hat{k}_6 - \hat{k}_{12}$ estimators based on the works of Hoerl and Kennard (1970a, 1970b), Nomura (1988), Kibria (2003), Khalaf and Shukur (2005), and Muniz and Kibria (2009), respectively.

$$\hat{k}_1 = \frac{1}{\hat{\alpha}^2_{max}}, \quad \hat{k}_2 = \frac{p+1}{\sum_{j=1}^{p+1} \frac{\hat{\alpha}_j^2}{(1+(1+\lambda_j(\hat{\alpha}_j^2)^{1/2}))}}, \quad \hat{k}_3 = \frac{1}{\left(\Pi_{j=1}^{p+1}\hat{\alpha}_j^2\right)^{1/(p+1)}},$$

$$\hat{k}_4 = median\left(\frac{1}{\hat{\alpha}_j^2}\right)_{j=1}^{p+1}, \quad \hat{k}_5 = \frac{\lambda_{max}}{(n-p-1)+\lambda_{max}\hat{\alpha}^2_{max}}, \quad \hat{k}_6 = \left(\Pi_{j=1}^{p+1}\frac{\lambda_j}{(n-p-1)+\lambda_j\hat{\alpha}_j^2}\right)^{1/(p+1)},$$

$$\hat{k}_7 = max\left(\frac{1}{\sqrt{\frac{1}{\hat{\alpha}_j^2}}}\right)_{j=1}^{p+1}, \quad \hat{k}_8 = max\left(\sqrt{\frac{1}{\hat{\alpha}_j^2}}\right)_{j=1}^{p+1}, \quad \hat{k}_9 = \left(\Pi_{j=1}^{p+1}\frac{1}{\sqrt{\frac{1}{\hat{\alpha}_j^2}}}\right)^{1/(p+1)},$$

$$\hat{k}_{10} = \left(\Pi_{j=1}^{p+1}\sqrt{\frac{1}{\hat{\alpha}_j^2}}\right)^{1/(p+1)}, \quad \hat{k}_{11} = median\left(\frac{1}{\sqrt{\frac{1}{\hat{\alpha}_j^2}}}\right)_{j=1}^{p+1}, \quad \hat{k}_{12} = median\left(\sqrt{\frac{1}{\hat{\alpha}_j^2}}\right)_{j=1}^{p+1}.$$

4. The Monte Carlo simulation study

In this section, we do a simulation study to compare the performance of the MLE and the PMQL ridge regression estimators based on twelve different ridge parameter estimation methods that are given in Subsec. 3.2. We use the SMSE criteria to compare the performance of estimators. We have followed the formula proposed by McDonald and Galarneau (1975) to generate the covariates with several degrees of multicollinearity. The formula is given as follows

$$x_{ij} = (1-\rho^2)^{1/2}m_{ij} + \rho m_{i,p+1}, i = 1,2\ldots,n, j = 1,2,\ldots,p, \tag{32}$$

where m_{ij}'s are independent standard normal pseudo-random numbers and ρ^2 represents the correlation between the covariates.

The response variable, y of the PMQL regression model is generated from the PMQL (μ_i, α, δ) by using the inverse transform method, where $\mu_i = \exp(x_i'\beta)$, $i = 1,2,\ldots,n$. The starting values of the slope parameters are selected such that $\sum_{j=1}^{p}\beta_j^2 = 1$ and $\beta_1 = \beta_2 = \ldots = \beta_p$ based on the work of Newhouse and Oman (1971).

The simulation study was designed based on the following factors.

(i) Three different values of ρ corresponding to 0.90, 0.95, and 0.99 are considered to examine the performance of the different estimators when increasing the degrees of correlation.
(ii) The asymptotic properties and the performance of the different estimators are examined by using four different sample sizes 10, 20, 30, and 40.
(iii) Following Månsson and Shukur (2011), we vary the value of the intercept β_0. When we decrease the value of the β_0, the average values of the $y_i(\mu_i)$, $i = 1,2,\ldots,n$ will decrease. This phenomenon leads to have more zeros of y which makes less variation in the sample and converging issue on the IWLS algorithm. Then, we choose two different values of the

intercept, β_0 corresponding to -1 and 1 to compare the performance of the different estimators.

(iv) p is taken to be 2 and 4 to examine the performance of the various estimators with the number of covariates.

(v) From Figure 1, we can observe that either changing the value of the over-dispersion parameter α or δ affects the variation of y. If there is an increment of variation y, it leads to have a negative impact on the performance of estimators (see Kibria 2003; Månsson 2012). Then, the values of α are taken to be 0.1 and 0.6, and the values of δ are taken to be 0.5 and 1.5 in order to compare the performance of different estimators.

The simulation is repeated 1000 times. To judge the performance of the different estimators, we obtain the SMSE values of different estimators by using the following equation:

$$\hat{\text{SMSE}}(\hat{\beta}) = \frac{\sum_{r=1}^{1000} (\hat{\beta}_r - \beta)'(\hat{\beta}_r - \beta)}{1000}, \tag{33}$$

where $\hat{\beta}_r$ is an estimator of β at the rth replication.

The results of Monte Carlo simulation study are summarized in Tables A1–A6 (Appendix). The minimum SMSE in each case is shown in bold. In general, it can be observed that:

(i) the PMQL ridge regression estimator performs better than MLE in the presence of multicollinearity.

(ii) the performances of MLE and PMQL ridge regression estimators based on different ridge parameter estimators are affected by the degrees of the correlation among the covariates, the sample size, the value of the intercept, the number of covariates, and the values of the over-dispersion parameters.

From the Tables A1–A6, when the degrees of the correlation increases we can note that:

(i) the SMSE of the MLE and the PMQL ridge regression estimator based on k_6 estimator increase.

(ii) in general, the SMSE of the PMQL ridge regression estimators based on k_2, k_7, k_{12} estimators decrease.

Further, the PMQL ridge regression estimators based on $k_2, k_7,$ and k_{12} estimators produce a smaller SMSE than those of other estimators reviewed in this study, and their SMSE are very close given a ρ value for all cases in general.

The asymptotic property holds for the MLE and most PMQL ridge regression estimators since the SMSE decreases or remains almost the same with the sample size. Further, in a given sample size, the PMQL ridge regression estimators based on $k_2, k_7,$ and k_{12} perform better than the MLE and other PMQL ridge regression estimators for all cases.

We can note that the decreasing value of the β_0 (1 to -1) leads to make an increment in the SMSE of the MLE, and the PMQL ridge regression estimators based on $k_1, k_5,$ and k_6 estimators. However, this change does not affect the SMSE of the rest of the estimators basically.

The increment of the number of covariates shows a negative impact on all given estimators in general and the proportion of times the MLE performs better than the PMQL ridge regression estimators decreases. Then, the benefit of the PMQL ridge regression is high when we have a higher number of covariates. In this situation also, the performance of the PMQL ridge regression estimators based on $k_2, k_7,$ and k_{12} estimators are better than the rest of the estimators reviewed in this study.

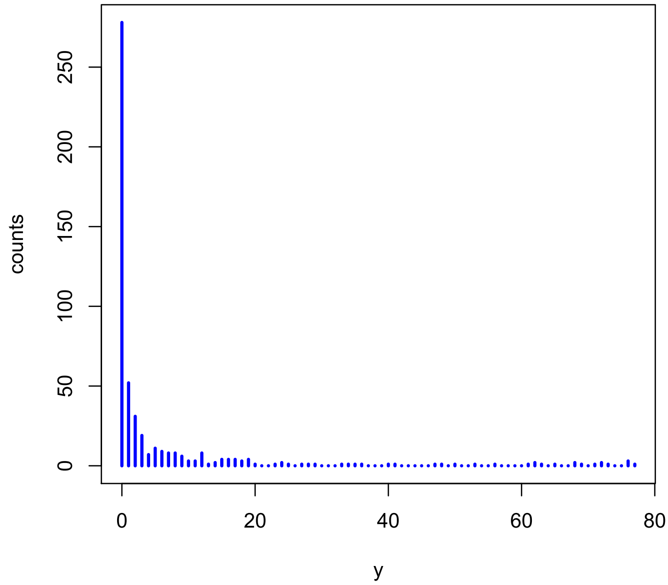

Figure 2. The distribution of the simulated response variable.

In general, as the value of α increases from 0.1 to 0.6, the SMSEs of the MLE and the PMQL ridge regression estimators become deflated, and as the value of δ increases from 0.5 to 1.5, the SMSEs also become deflated. From Figure 1, we can note that in both situations, the variance of y decreases. It leads to make a positive impact on the performance of estimators.

Hence, among the reviewed ridge parameter estimators in this study, none of the estimated value of k always shows a better performance than other estimated k values for all cases in general. However, in general, we may conclude that the ridge parameter estimators, $k_2, k_7,$ and k_{12} are the best options to estimate the ridge parameter, k of $RE_{(PMQL)}$ for all cases.

5. Applications

In this section, we provide a simulated data and a real-world applications in order to show the performance of the PMQL ridge regression estimator over the $MLE_{(PMQL)}$.

5.1. Simulated data application

A data set with $\rho = 0.9999,\ p = 4,\quad n = 500,\quad \beta_0 = 1,\quad \alpha = 0.60,$ and $\delta = 0.25$ is simulated by using the method discussed in Sec. 4. The skewness, excess kurtosis, and index of dispersion of the simulated response variable y are 3.572, 9.759, and 35.381, respectively. Then it is clear that the y has higher positive skewness, a long-right tail, and higher over-dispersion. The distribution of y is illustrated in Figure 2.

Table 1 shows the estimated regression coefficients, their standard errors (SEs) (in parentheses), and SMSE values of the different PMQL ridge estimators and the $MLE_{(PMQL)}$. From Table 1 results, it is clear that the PMQL ridge regression estimators perform better than the $MLE_{(PMQL)}$ providing smaller SEs and SMSE values compared with $MLE_{(PMQL)}$. Further, among the PMQL ridge regression estimators, the PMQL ridge regression estimator based on the ridge parameter estimator k_{12} shows a better performance than others.

Table 1. The estimated regression coefficients, standard errors (in parentheses) and SMSEs of the MLE and the PMQL ridge regression estimators for the simulated data.

Parameter	$MLE_{(PMQL)}$(SE)	$RE_{(PMQL)}$(SE)		
		k_2	k_7	k_{12}
$\hat{\beta}_0$	0.151 (0.121)	0.137 (0.187)	0.150 (0.119)	0.147 (0.111)
$\hat{\beta}_1$	0.261 (7.429)	0.489 (0.298)	0.538 (0.097)	0.530 (0.036)
$\hat{\beta}_2$	0.917 (7.417)	0.489 (0.298)	0.544 (0.097)	0.531 (0.036)
$\hat{\beta}_3$	0.687 (6.941)	0.489 (0.298)	0.542 (0.102)	0.530 (0.037)
$\hat{\beta}_4$	0.307 (7.029)	0.490 (0.299)	0.538 (0.101)	0.530 (0.037)
SMSE	207.837	0.080	0.054	**0.023**

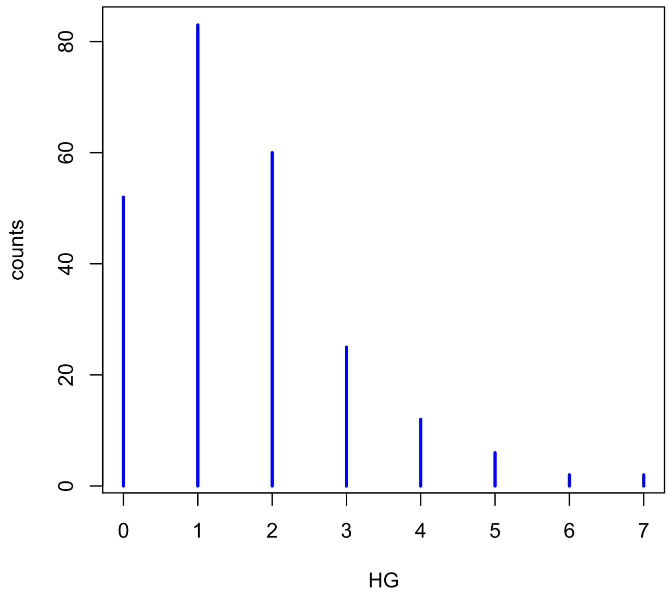

Figure 3. The distribution of the response variable HG.

Table 2. Bivariate correlations among the covariates.

Covariates	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1.000	−0.688	**0.996**	−0.687	**0.995**	−0.721
x_2		1.000	−0.669	**0.994**	−0.679	**0.990**
x_3			1.000	−0.670	**0.998**	−0.703
x_4				1.000	−0.681	**0.994**
x_5					1.000	−0.714
x_6						1.000

5.2. Real-world application

In this section, we illustrate the applicability of the PMQL ridge estimator by using the Swedish football data set which includes the Swedish football teams' performance in the top Swedish league (Allsvenskan) during the year 2012. The data set is publicly available at http://www.football-data.co.uk/sweden.php. Here, we try to explain the number of full-time home team goals (HG) with six covariates by fitting the PMQL regression model. This data set contains 242 observations. The covariates are the pinnacle home win odds (x_1), pinnacle away win odds (x_2), maximum odds portal home win (x_3), maximum odds portal away win (x_4), average odds portal home win (x_5), and average odds portal away win (x_6). Qasim et al. (2019) used a similar data set to fit a Poisson regression model in which the response variable HG is not over-dispersed.

Table 3. The estimated regression coefficients, standard errors (in parentheses) and SMSEs of the MLE and the PMQL ridge regression estimators.

Parameter	$MLE_{(PMQL)}$(SE)	$RE_{(PMQL)}$(SE)		
		k_2	k_7	k_{12}
$\hat{\beta}_0$	−1.046 (0.180)	−0.307 (0.044)	−1.013(0.166)	−0.939 (0.144)
$\hat{\beta}_1$	0.408 (0.236)	0.328 (0.026)	0.397 (0.218)	0.377 (0.181)
$\hat{\beta}_2$	0.408 (0.076)	0.438 (0.040)	0.410 (0.075)	0.413 (0.072)
$\hat{\beta}_3$	0.407 (0.351)	0.437 (0.021)	0.438 (0.289)	0.482 (0.198)
$\hat{\beta}_4$	0.409 (0.090)	0.492 (0.038)	0.414 (0.088)	0.427 (0.084)
$\hat{\beta}_5$	0.408 (0.368)	0.297 (0.021)	0.377 (0.306)	0.335 (0.212)
$\hat{\beta}_6$	0.406 (0.101)	0.194 (0.040)	0.395 (0.099)	0.367 (0.093)
SMSE	0.371	**0.026**	0.276	0.160

However, the variable HG is over-dispersed (variance to mean ratio equals $1.201 > 1$) in the data set that we use here. Figure 3 illustrates the distribution of the variable HG.

Table 2 displays the bivariate correlations among the covariates. It is clear that the bivariate correlations between x_1 and x_3, x_1 and x_5, x_2 and x_4, x_2 and x_6, x_3 and x_5, x_4 and x_6 are very high and they are greater than 0.99. Further, the conditional number (ratio of the maximum to minimum eigenvalues) is 33,460.35 which is much larger than 1000. These results indicate that there is severe multicollinearity among the covariates in this data set (see Türkan and Özel 2016). To examine whether the PMQL regression model is suitable for the response variable HG, the Chi-square (χ^2) goodness of fit test is used. The χ^2 value is computed as 3.159 with p-value equals 0.531. Then, this test confirms that the PMQL regression model fits well for this data set.

The estimated regression coefficients, their standard errors (SEs) (in parentheses), and SMSE values for the MLE and PMQL ridge estimators are listed in Table 3. We noted that the PMQL ridge regression estimators having the estimated k values; $k_2, k_7,$ and k_{12} performed well in the simulation study. Therefore, the same ridge parameter estimators are also used in this application to estimate the k in the PMQL ridge regression estimator. From Table 3, we can observe that the PMQL ridge estimators perform better than the MLE in the SMSE sense. Further, the PMQL ridge regression estimator based on k_2 estimator has the minimum SMSE.

6. Conclusion

This paper adopts the ridge regression estimator as an alternative method to the maximum likelihood estimator (MLE) to combat the multicollinearity problem to estimate the coefficients of a newly introduced mixed Poisson regression model namely, the PMQL regression model. The scalar mean square error (SMSE) properties of the PMQL ridge regression estimator are discussed and the possible estimators of its ridge parameter are obtained. A Monte Carlo simulation study is conducted to evaluate the performance of MLE and the PMQL ridge regression estimators based on the different shrinkage parameter estimation methods of its ridge parameter in the SMSE sense. The simulation study results show that the PMQL ridge regression estimators perform better than the traditional MLE in the presence of multicollinearity among the covariates. The performances of the different estimators of the regression coefficients are affected by the factors such as the degrees of correlation among the covariates (ρ), the sample size (n), the intercept (β_0), the number of covariates (p), and the over-dispersion parameters of the PMQL regression model (α, δ). Further, it is observed that the PMQL ridge estimators based on the ridge parameter estimators, $k_2, k_7,$ and k_{12} produce smaller SMSE basically in all different situations. The results of the simulated and real-world applications are also consistent with the results of the simulation study. Therefore, by considering the results of this study, the PMQL ridge regression estimator based on the ridge parameter estimators, $k_2, k_7,$ and k_{12} can be recommended to analyze the over-dispersed count responses when multicollinearity exists among the covariates.

Appendix: Results of the simulation study

Table A1. Estimated SMSE values for different ρ, n, p, k when $\beta_0 = -1, \delta = 0.5, \alpha = 0.1$.

	$\rho = 0.90$				$\rho = 0.95$				$\rho = 0.99$			
Estimator	$n=10$	$n=20$	$n=30$	$n=40$	$n=10$	$n=20$	$n=30$	$n=40$	$n=10$	$n=20$	$n=30$	$n=40$
$p=2$												
MLE	5.777	2.107	1.304	0.935	9.937	3.572	2.198	1.573	43.417	15.358	9.389	6.699
k_1	1.752	1.229	0.921	0.741	1.771	1.462	1.206	1.020	1.331	1.359	1.413	1.438
k_2	0.601	**0.376**	**0.288**	**0.241**	0.557	**0.347**	**0.267**	**0.224**	0.511	0.310	0.238	0.201
k_3	0.983	0.857	0.798	0.792	1.060	0.988	0.939	0.959	1.093	0.946	0.869	0.885
k_4	0.911	0.745	0.683	0.647	0.947	0.843	0.817	0.799	0.859	0.751	0.727	0.731
k_5	2.244	1.433	1.022	0.785	2.335	1.781	1.393	1.126	1.735	1.845	1.889	1.856
k_6	4.425	1.909	1.227	0.895	6.790	3.082	2.005	1.472	19.490	10.864	7.522	5.702
k_7	**0.581**	0.383	0.316	0.279	**0.546**	0.348	0.287	0.256	**0.509**	**0.301**	**0.232**	**0.197**
k_8	0.930	0.725	0.615	0.558	0.991	0.801	0.667	0.638	1.070	0.795	0.636	0.592
k_9	1.463	1.155	0.912	0.738	1.693	1.550	1.292	1.104	1.603	1.721	1.767	1.896
k_{10}	0.695	0.470	0.378	0.318	0.680	0.472	0.372	0.332	0.672	0.434	0.333	0.293
k_{11}	1.251	1.050	0.862	0.711	1.396	1.347	1.200	1.038	1.135	1.319	1.427	1.580
k_{12}	0.681	0.450	0.356	0.291	0.639	0.431	0.342	0.285	0.583	0.380	0.289	0.244
$p=4$												
MLE	28.927	6.560	3.590	2.469	55.700	12.254	6.666	4.567	270.623	57.845	31.313	21.377
k_1	3.183	2.913	2.296	1.828	3.388	3.620	3.160	2.689	2.321	3.181	3.650	3.828
k_2	**0.616**	**0.424**	**0.346**	**0.306**	0.561	**0.389**	**0.323**	**0.289**	0.499	0.337	0.282	0.256
k_3	0.960	0.782	0.694	0.664	1.070	0.967	0.903	0.863	0.932	0.812	0.750	0.697
k_4	0.851	0.657	0.557	0.516	0.935	0.796	0.722	0.701	0.826	0.649	0.583	0.549
k_5	3.839	3.287	2.469	1.921	4.063	4.164	3.487	2.896	2.794	3.929	4.414	4.539
k_6	12.309	5.343	3.229	2.299	20.882	9.572	5.860	4.186	75.119	40.246	25.768	18.716
k_7	0.636	0.528	0.499	0.482	**0.546**	0.433	0.416	0.416	**0.461**	**0.310**	**0.265**	0.248
k_8	1.008	0.771	0.656	0.612	1.054	0.868	0.769	0.693	0.990	0.783	0.673	0.592
k_9	2.888	2.785	2.255	1.829	3.505	4.027	3.585	3.028	2.236	3.526	4.394	4.663
k_{10}	0.673	0.481	0.393	0.352	0.644	0.439	0.352	0.303	0.585	0.375	0.290	0.239
k_{11}	2.346	2.441	2.061	1.710	2.758	3.406	3.175	2.797	1.873	2.602	3.302	3.769
k_{12}	0.679	0.493	0.412	0.371	0.611	0.419	0.342	0.299	0.564	0.346	0.269	**0.236**

Table A2. Estimated SMSE values for different ρ, n, p, k when $\beta_0 = 1, \delta = 0.5, \alpha = 0.1$.

	$\rho = 0.90$				$\rho = 0.95$				$\rho = 0.99$			
Estimator	$n=10$	$n=20$	$n=30$	$n=40$	$n=10$	$n=20$	$n=30$	$n=40$	$n=10$	$n=20$	$n=30$	$n=40$
$p=2$												
MLE	2.782	1.075	0.674	0.487	4.848	1.859	1.163	0.838	21.460	8.168	5.094	3.665
k_1	1.216	1.263	1.301	1.350	1.216	1.319	1.390	1.433	1.117	1.121	1.162	1.221
k_2	0.900	0.785	0.744	0.726	0.891	0.790	0.755	0.739	0.817	0.732	0.700	0.693
k_3	1.327	1.220	1.165	1.129	1.338	1.279	1.229	1.204	1.172	1.083	1.025	1.023
k_4	0.917	0.751	0.671	0.624	0.918	0.786	0.718	0.669	0.790	0.681	0.613	0.605
k_5	0.773	0.551	0.424	0.345	0.725	0.583	0.490	0.425	0.516	0.443	0.420	0.420
k_6	2.012	0.949	0.624	0.460	3.056	1.545	1.035	0.771	8.247	5.343	3.874	3.000
k_7	0.571	0.412	0.333	0.281	**0.516**	0.394	0.344	0.309	**0.411**	**0.278**	0.245	0.235
k_8	0.977	0.776	0.674	0.600	0.979	0.801	0.695	0.629	0.860	0.655	0.552	0.515
k_9	1.486	0.867	0.599	0.451	1.800	1.274	0.934	0.727	1.726	1.980	1.926	1.836
k_{10}	0.706	0.481	0.379	0.313	0.697	0.485	0.381	0.327	0.607	0.401	0.314	0.278
k_{11}	1.137	0.748	0.541	0.419	1.260	1.007	0.793	0.642	0.986	1.220	1.289	1.323
k_{12}	**0.569**	**0.369**	**0.282**	**0.235**	0.543	**0.354**	**0.277**	**0.232**	0.470	0.292	**0.219**	**0.190**
$p=4$												
MLE	12.599	3.415	1.954	1.370	24.123	6.463	3.688	2.582	116.568	30.875	17.584	12.295
k_1	1.204	1.808	1.304	1.322	1.125	1.186	1.207	1.228	0.906	0.809	0.790	0.823
k_2	1.170	1.160	1.160	1.161	1.083	1.093	1.101	1.103	0.761	0.762	0.758	0.753
k_3	1.577	1.601	1.608	1.620	1.506	1.537	1.559	1.552	1.124	1.132	1.138	1.158
k_4	1.319	1.360	1.358	1.373	1.236	1.276	1.280	1.283	0.817	0.811	0.795	0.790
k_5	0.826	0.771	0.700	0.630	0.710	0.701	0.694	0.677	0.565	0.446	0.449	0.484
k_6	5.716	2.787	1.754	1.273	9.807	5.069	3.237	2.362	35.823	21.644	14.454	10.740
k_7	0.847	0.792	0.716	0.638	0.707	0.730	0.736	0.716	0.537	0.405	0.441	0.484
k_8	1.253	1.079	0.988	0.950	1.209	1.043	0.978	0.886	1.059	0.867	0.765	0.737
k_9	4.872	2.803	1.799	1.307	5.716	4.426	3.117	2.331	3.547	6.061	6.620	6.392
k_{10}	0.773	0.587	0.495	0.441	0.710	0.541	0.459	0.396	0.529	0.366	0.294	0.258
k_{11}	3.887	2.605	1.737	1.279	4.276	3.882	2.898	2.231	2.102	4.086	4.836	4.895
k_{12}	**0.658**	**0.480**	**0.389**	**0.338**	**0.602**	**0.434**	**0.355**	**0.305**	**0.451**	**0.301**	**0.235**	**0.202**

Table A3. Estimated SMSE values for different ρ, n, p, k when $\beta_0 = -1, \delta = 1.5, \alpha = 0.1$.

	$\rho = 0.90$				$\rho = 0.95$				$\rho = 0.99$			
Estimator	$n=10$	$n=20$	$n=30$	$n=40$	$n=10$	$n=20$	$n=30$	$n=40$	$n=10$	$n=20$	$n=30$	$n=40$
$p=2$												
MLE	3.909	1.348	0.819	0.582	6.648	2.246	1.354	0.958	28.724	9.475	5.655	3.986
k_1	1.477	0.919	0.660	0.505	1.628	1.193	0.919	0.738	1.356	1.452	1.463	1.406
k_2	0.464	**0.273**	**0.204**	**0.167**	0.423	**0.248**	**0.187**	**0.156**	0.383	**0.215**	**0.161**	**0.134**
k_3	0.726	0.564	0.520	0.478	0.749	0.638	0.587	0.559	0.694	0.499	0.458	0.426
k_4	0.674	0.485	0.420	0.378	0.661	0.519	0.479	0.440	0.565	0.400	0.357	0.334
k_{K5}	1.699	0.986	0.682	0.514	1.899	1.297	0.966	0.757	1.587	1.677	1.629	1.539
k_6	2.868	1.199	0.762	0.552	4.301	1.891	1.216	0.887	11.722	6.355	4.369	3.300
k_7	**0.439**	0.303	0.257	0.227	**0.401**	0.274	0.238	0.218	**0.357**	0.220	0.180	0.160
k_8	0.676	0.474	0.397	0.338	0.699	0.503	0.407	0.360	0.704	0.424	0.345	0.291
k_9	1.156	0.831	0.622	0.488	1.331	1.117	0.885	0.725	1.084	1.130	1.231	1.229
k_{10}	0.528	0.341	0.266	0.221	0.489	0.309	0.246	0.205	0.435	0.254	0.203	0.172
k_{11}	0.991	0.778	0.600	0.478	1.114	1.016	0.848	0.703	0.831	0.937	1.049	1.112
k_{12}	0.571	0.616	0.433	0.335	0.958	0.573	0.422	0.339	0.802	0.524	0.398	0.329
$p=4$												
MLE	20.509	4.002	2.096	1.410	39.475	7.369	3.821	2.555	191.737	34.362	17.656	11.739
k_1	2.765	2.137	1.537	1.149	3.142	2.963	2.331	1.829	2.688	3.553	3.734	3.643
k_2	**0.489**	**0.330**	**0.268**	**0.235**	**0.438**	**0.295**	**0.247**	**0.222**	0.404	**0.246**	**0.205**	**0.187**
k_3	0.698	0.487	0.395	0.345	0.689	0.534	0.440	0.416	0.556	0.397	0.345	0.324
k_4	0.655	0.452	0.364	0.318	0.618	0.442	0.375	0.337	0.523	0.335	0.271	0.256
k_5	3.054	2.261	1.579	1.172	3.444	3.136	2.411	1.873	2.937	3.872	4.001	3.861
k_6	7.877	3.187	1.866	1.305	13.173	5.605	3.320	2.326	46.368	23.029	14.286	10.175
k_7	0.567	0.515	0.483	0.449	0.467	0.424	0.425	0.424	**0.350**	0.268	0.255	0.259
k_8	0.745	0.520	0.426	0.366	0.716	0.553	0.437	0.411	0.636	0.447	0.377	0.350
k_9	2.368	1.873	1.401	1.081	2.576	2.695	2.202	1.807	1.527	2.365	2.907	3.143
k_{10}	0.576	0.457	0.402	0.363	0.496	0.356	0.308	0.274	0.416	0.277	0.226	0.203
k_{11}	1.911	1.661	1.301	1.032	2.092	2.283	1.991	1.661	1.238	1.815	2.229	2.511
k_{12}	0.613	0.507	0.449	0.402	0.510	0.389	0.347	0.319	0.429	0.289	0.248	0.229

Table A4. Estimated SMSE values for different ρ, n, p, k when $\beta_0 = 1, \delta = 1.5, \alpha = 0.1$.

	$\rho = 0.90$				$\rho = 0.95$				$\rho = 0.99$			
Estimator	$n=10$	$n=20$	$n=30$	$n=40$	$n=10$	$n=20$	$n=30$	$n=40$	$n=10$	$n=20$	$n=30$	$n=40$
$p=2$												
MLE	1.273	0.481	0.300	0.216	2.208	0.825	0.513	0.368	9.722	3.591	2.223	1.595
k_1	0.983	0.934	0.937	0.983	1.020	1.007	1.002	1.050	0.809	0.833	0.832	0.857
k_2	0.653	0.599	0.584	0.584	0.635	0.594	0.583	0.586	0.520	0.478	0.461	0.465
k_3	0.954	0.839	0.750	0.726	0.940	0.862	0.817	0.800	0.750	0.677	0.620	0.619
k_4	0.645	0.508	0.436	0.407	0.620	0.534	0.460	0.450	0.477	0.387	0.341	0.324
k_5	0.420	0.253	0.190	0.152	0.406	0.276	0.222	0.186	0.334	0.255	0.220	0.201
k_6	0.919	0.424	0.277	0.204	1.389	0.685	0.456	0.339	3.718	2.345	1.688	1.303
k_7	0.390	0.262	0.199	0.159	0.366	0.286	0.239	0.204	0.265	0.219	0.216	0.218
k_8	0.614	0.425	0.329	0.285	0.587	0.425	0.360	0.312	0.504	0.352	0.275	0.248
k_9	0.872	0.431	0.283	0.208	1.167	0.673	0.459	0.344	1.464	1.446	1.244	1.086
k_{10}	0.397	0.234	0.171	0.139	0.367	0.233	0.177	0.150	0.304	0.182	0.134	0.114
k_{11}	0.756	0.406	0.273	0.204	0.928	0.607	0.430	0.330	0.914	1.036	0.981	0.901
k_{12}	**0.347**	**0.202**	**0.148**	**0.119**	**0.316**	**0.199**	**0.146**	**0.126**	**0.259**	**0.156**	**0.119**	**0.099**
$p=4$												
MLE	6.094	1.529	0.858	0.597	11.719	2.881	1.610	1.117	56.853	13.700	7.635	5.285
k_1	0.801	0.676	0.641	0.584	0.792	0.612	0.527	0.484	0.954	0.558	0.413	0.357
k_2	0.760	0.780	0.792	0.806	0.650	0.647	0.653	0.662	0.413	0.341	0.305	0.304
k_3	1.007	0.978	0.975	0.952	0.899	0.841	0.817	0.781	0.566	0.504	0.440	0.408
k_4	0.754	0.697	0.696	0.660	0.669	0.595	0.573	0.538	0.439	0.364	0.300	0.274
k_5	0.542	0.391	0.314	0.275	0.518	0.373	0.333	0.312	0.798	0.431	0.369	0.325
k_6	2.685	1.244	0.733	0.554	4.591	2.252	1.413	1.022	16.682	9.558	6.272	4.617
k_7	0.706	0.598	0.770	0.394	0.631	0.652	0.599	0.534	0.344	0.437	0.518	0.570
k_8	0.771	0.593	0.482	0.465	0.709	0.530	0.448	0.381	0.573	0.398	0.305	0.259
k_9	2.811	1.332	0.514	0.578	3.380	2.151	1.417	1.034	2.697	3.697	3.475	3.084
k_{10}	0.413	0.269	0.813	0.178	0.364	0.240	0.192	0.170	0.274	0.281	0.244	0.174
k_{11}	2.384	1.273	0.213	0.570	2.696	1.971	1.355	1.006	1.734	2.763	2.836	2.662
k_{12}	**0.397**	**0.255**	**0.196**	**0.169**	**0.358**	**0.237**	**0.191**	**0.164**	**0.262**	**0.214**	**0.176**	**0.156**

Table A5. Estimated SMSE values for different ρ, n, p, k when $\beta_0 = -1, \delta = 0.5, \alpha = 0.6$.

	$\rho = 0.90$				$\rho = 0.95$				$\rho = 0.99$			
Estimator	$n = 10$	$n = 20$	$n = 30$	$n = 40$	$n = 10$	$n = 20$	$n = 30$	$n = 40$	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$p = 2$												
MLE	5.518	2.003	1.238	0.887	9.481	3.390	2.083	1.489	41.383	14.552	8.880	6.330
k_1	1.733	1.179	0.894	0.703	1.753	1.435	1.186	0.987	1.339	1.379	1.431	1.431
k_2	1.583	**0.363**	**0.277**	**0.231**	0.540	**0.333**	**0.256**	**0.215**	0.495	0.297	0.227	**0.191**
k_3	0.950	0.828	0.773	0.750	1.021	0.945	0.904	0.909	1.005	0.868	0.801	0.815
k_4	0.879	0.723	0.659	0.618	0.900	0.811	0.783	0.764	0.801	0.691	0.676	0.678
k_5	2.186	1.364	0.979	0.749	2.281	1.721	1.342	1.078	1.726	1.827	1.870	1.798
k_6	4.208	1.811	1.163	0.849	6.442	2.918	1.897	1.392	18.388	10.236	7.087	5.373
k_7	**0.561**	0.371	0.308	0.273	**0.526**	0.337	0.279	0.250	**0.489**	**0.289**	**0.224**	0.192
k_8	0.899	0.700	0.588	0.522	0.961	0.758	0.645	0.591	0.996	0.736	0.587	0.538
k_9	1.439	1.121	0.876	0.706	1.674	1.496	1.240	1.052	1.405	1.597	1.660	1.790
k_{10}	0.674	0.455	0.364	0.298	0.657	0.449	0.358	0.308	0.627	0.407	0.307	0.273
k_{11}	1.219	1.025	0.832	0.683	1.350	1.322	1.160	1.000	1.042	1.224	1.381	1.531
k_{12}	0.658	0.439	0.340	0.276	0.614	0.418	0.323	0.272	0.557	0.357	0.271	0.232
$p = 4$												
MLE	27.764	6.213	3.388	2.326	53.464	11.591	6.282	4.296	259.773	54.662	29.469	20.078
k_1	3.152	2.846	2.213	1.744	3.363	3.567	3.072	2.595	2.336	3.212	3.665	3.827
k_2	**0.600**	**0.411**	**0.335**	**0.296**	0.543	**0.375**	**0.312**	**0.280**	0.480	0.321	0.270	0.245
k_3	0.933	0.733	0.648	0.604	1.013	0.890	0.821	0.807	0.853	0.727	0.658	0.629
k_4	0.828	0.621	0.527	0.476	0.872	0.730	0.667	0.637	0.742	0.572	0.504	0.483
k_5	3.748	3.175	2.361	1.824	3.983	4.055	3.356	2.769	2.772	3.898	4.359	4.464
k_6	11.697	5.049	3.044	2.165	19.817	9.031	5.515	3.935	71.158	37.895	24.208	17.56
k_7	0.625	0.525	0.497	0.480	**0.534**	0.430	0.416	0.417	**0.445**	**0.303**	0.262	0.248
k_8	0.984	0.739	0.627	0.567	1.011	0.810	0.701	0.673	0.933	0.720	0.620	0.556
k_9	2.885	2.658	2.150	1.727	3.321	3.796	3.358	2.869	2.032	3.233	3.958	4.416
k_{10}	0.662	0.468	0.389	0.347	0.616	0.418	0.337	0.297	0.544	0.349	0.269	0.229
k_{11}	2.296	2.337	1.965	1.618	2.595	3.212	3.011	2.635	1.656	2.349	2.971	3.511
k_{12}	0.670	0.488	0.415	0.372	0.589	0.407	0.335	0.297	0.526	0.328	**0.256**	**0.224**

Table A6. Estimated SMSE values for different ρ, n, p, k when $\beta_0 = 1, \delta = 0.5, \alpha = 0.6$.

	$\rho = 0.90$				$\rho = 0.95$				$\rho = 0.99$			
Estimator	$n = 10$	$n = 20$	$n = 30$	$n = 40$	$n = 10$	$n = 20$	$n = 30$	$n = 40$	$n = 10$	$n = 20$	$n = 30$	$n = 40$
$p = 2$												
MLE	2.566	0.990	0.621	0.448	4.471	1.712	1.070	0.771	19.784	7.516	4.685	3.371
k_1	1.191	1.228	1.276	1.328	1.230	1.288	1.327	1.380	1.036	1.067	1.085	1.135
k_2	0.870	0.765	0.730	0.714	0.861	0.769	0.737	0.724	0.771	0.696	0.667	0.657
k_3	1.285	1.186	1.142	1.097	1.304	1.244	1.187	1.173	1.097	1.024	0.974	0.949
k_4	0.903	0.758	0.678	0.633	0.904	0.787	0.730	0.682	0.736	0.653	0.599	0.583
k_5	0.730	0.512	0.391	0.316	0.679	0.543	0.454	0.391	0.496	0.416	0.395	0.383
k_6	1.854	0.874	0.575	0.424	2.815	1.422	0.953	0.709	7.588	4.909	3.559	2.756
k_7	0.546	0.392	0.316	0.265	**0.495**	0.379	0.330	0.297	**0.390**	**0.267**	0.238	0.230
k_8	0.928	0.723	0.633	0.553	0.935	0.760	0.644	0.594	0.799	0.597	0.507	0.455
k_9	1.396	0.811	0.557	0.417	1.726	1.199	0.874	0.676	1.620	1.898	1.841	1.720
k_{10}	0.662	0.449	0.354	0.292	0.654	0.457	0.360	0.305	0.552	0.362	0.286	0.249
k_{11}	1.101	0.714	0.511	0.393	1.235	0.979	0.761	0.609	0.936	1.211	1.292	1.294
k_{12}	**0.541**	**0.352**	**0.266**	**0.221**	0.515	**0.335**	**0.264**	**0.218**	0.433	0.270	**0.207**	**0.175**
$p = 4$												
MLE	11.680	3.149	1.799	1.261	22.374	5.959	3.396	2.376	108.162	28.462	16.185	11.309
k_1	1.183	1.188	1.194	1.202	1.062	1.096	1.081	1.092	0.872	0.709	0.690	0.734
k_2	1.121	1.116	1.121	1.123	1.018	1.039	1.045	1.051	0.700	0.683	0.665	0.670
k_3	1.518	1.530	1.554	1.565	1.423	1.444	1.472	1.480	1.030	1.033	1.020	1.031
k_4	1.257	1.277	1.297	1.293	1.132	1.185	1.193	1.188	0.755	0.714	0.683	0.706
k_5	0.781	0.718	0.652	0.579	0.685	0.661	0.643	0.629	0.586	0.448	0.437	0.456
k_6	5.288	2.569	1.615	1.172	9.073	4.672	2.980	2.120	33.134	19.941	13.301	9.877
k_7	0.828	0.771	0.691	0.611	0.693	0.722	0.723	0.699	**0.422**	0.403	0.444	0.490
k_8	1.196	1.005	0.924	0.888	1.147	0.961	0.895	0.831	0.979	0.789	0.705	0.645
k_9	4.560	2.587	1.659	1.204	5.247	4.054	2.864	2.147	3.285	5.627	6.003	5.767
k_{10}	0.720	0.524	0.444	0.392	0.649	0.482	0.405	0.351	0.482	0.327	0.255	0.218
k_{11}	3.689	2.413	1.606	1.180	3.911	3.595	2.673	2.056	2.050	3.738	4.372	4.530
k_{12}	**0.620**	**0.434**	**0.354**	**0.304**	**0.553**	**0.393**	**0.318**	**0.270**	0.425	**0.276**	**0.221**	**0.183**

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