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Squeezed States in the Quaternionic Setting

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Abstract

Using a left multiplication defined on a right quaternionic Hilbert space, we shall demonstrate that pure squeezed states, which are obtained by the sole action of the squeeze operator on the vacuum state, can be defined with all the desired properties on a right quaternionic Hilbert space. Further, we shall also demonstrate that squeezed states, which are obtained by the action of the squeeze operator on canonical coherent states, in other words they are obtained by the action of the displacement operator followed by the action of the squeeze operator on the vacuum state, can be defined on the same Hilbert space, but the non-commutativity of quaternions prevents us in getting the desired results. However, we will show that if one considers the quaternionic slice wise approach, then the desired properties can be obtained for quaternionic squeezed states.

Keywords Quaternion · Displacement operator · Squeezed operator · Coherent states · Lie algebra

Mathematics Subject Classification (2010) Primary 81R30 · 46E22

1 Introduction

As it is well known, quantum mechanics can be formulated over the complex and the quaternionic numbers, see [7]. In recent times, new mathematical tools in quaternionic analysis became available in the literature and, as a consequence, there has

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been a resurgence of interest for the quaternionic quantum mechanics. In this formulation, in complete analogy with the complex formulation, states are represented by vectors of a separable quaternionic Hilbert space and observables are represented by quaternionic linear and self-adjoint operators, see, for example, the celebrated book [1] for more information.

However, until the most recent times, an appropriate spectral theory was missing since there was not a satisfactory notion of spectrum in quaternionic functional calculus. This difficulty has been solved with the introduction of the so-called S-spectrum (see [9]) and, accordingly, with a proof of the spectral theorem for normal operators, see [4].

In a right quaternionic Hilbert space with a right multiplication on it, in general, for any quaternionic linear operator A and $\mathfrak{q} \in \mathbb{H}$, the quaternions, $(\mathfrak{q}A)^\dagger \neq \overline{\mathfrak{q}}A^\dagger$. Due to this we cannot define a linear self-adjoint operator in quaternionic quantum mechanics similar to the complex momentum operator [17]. For various attempt in defining a quaternionic self-adjoint momentum operator and their drawbacks see [1]. However, in our paper [18], we offered a discussion on the various notions of momentum operator and we show that, by using the notion of left multiplication in a right quaternionic Hilbert space it is possible to define a linear self-adjoint momentum operator in complete analogy with the complex case. The possibility to introduce a left multiplication in a right quaternionic Hilbert space is very well known and a very useful tool in several cases. In fact a linear space over the quaternions is, in general, one sided (either left or right). However, in order to have good properties when considering linear operators acting on the space, it is necessary to have a multiplication on both sides. It can always be defined but it requires to fix a Hilbert basis. Therefore the results obtained with such a choice of basis have to be shown to be independent of the choice of the basis. However, when we consider a particular quantum system we always work with a fixed basis, which is the wavefunctions of the Hamiltonian (Fock space basis). Therefore we do not need to be concerned about working with a fixed basis.

In [18] we have also deepened the study of an appropriate harmonic oscillator displacement operator showing that this displacement operator leads to a square integrable, irreducible and unitary representation and that it satisfies most of the properties of its complex counterpart.

In this paper we introduce and study the squeeze operator which is formally defined as in the complex setting but where the operation involved in the definition have to be interpreted in an appropriate way. To be specific, a squeeze operator is obtained by exponentiating $\frac{1}{2}(\mathfrak{p} \cdot (\mathsf{a}^\dagger)^2 - \overline{\mathfrak{p}} \cdot \mathsf{a}^2)$ where $\mathfrak{p}\cdot$ is the left multiplication by the quaternion $\mathfrak{p}$ and $\mathsf{a}^\dagger$, a are the creation and annihilation operators respectively. We show that this latter operator is anti-hermitian and we study several properties of the squeezed operator. We also study quaternionic pure squeezed states, obtained by the action of the squeeze operator on the vacuum state. For clarity, as in the complex case, the distinction between pure squeezed states, squeezed states, and the two photon states is as follows [10, 14, 22].

- Pure squeezed states: $S(\mathfrak{p})\Phi_0$.
- Squeezed states: $S(\mathfrak{p})\mathfrak{D}(\mathfrak{q})\Phi_0$.

- Two photons states: $\mathfrak{D}(\mathfrak{q})S(\mathfrak{p})\Phi_0$,

where $\mathfrak{D}(\mathfrak{q})$ is the displacement operator, $S(\mathfrak{p})$ is the squeeze operator, and Φ_0 is the vacuum state of the Fock space.

Due to the non-commutative nature of quaternions, there is an intrinsic issue if one is aimed to obtain relations involving both the displacement and the squeeze operator. Suitable relations can be obtained only slice-wise.

There is a vast interest in squeezed states in various applications, particularly in the coding and transmission of information through optical devices [10, 14, 22]. In the quaternion case, these squeezed states appear as two component states in four variables. Hence these states have more degrees of freedom and may be useful in application.

The plan of the paper is as follows. Section 2 contains some preliminaries on quaternions, right quaternionic Hilbert spaces and the notion of left multiplication. Section 3 studies the Bargmann space of regular functions, the displacement operator, the squeeze operator and some of its properties. We also introduce some quaternionic Lie algebras constructed by taking some suitable real or complex linear spaces and equipping them with suitable Lie brackets. The expectation values and the variances of the creation and annihilation operator and of the quadrature operators are computed with pure squeezed states in this section. We also define the squeezed states by consecutively applying the displacement operator and the squeeze operator. The fourth section is devoted to the study of squeezed states on a quaternion slice. We also prove a disentanglement formula and obtain the squeezed basis in terms of the quaternionic Hermite polynomials. Section 5 ends the manuscript with a conclusion.

2 Mathematical Preliminaries

In this section we recall some basic facts about quaternions, their complex matrix representation, quaternionic Hilbert spaces as needed here. For details we refer the reader to [1, 6, 21, 23].

2.1 Quaternions

Let $\mathbb{H}$ denote the field of quaternions. Its elements are of the form $\mathfrak{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ where q_0, q_1, q_2 and q_3 are real numbers, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are imaginary units such that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$ and $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. The quaternionic conjugate of $\mathfrak{q}$ is defined to be $\overline{\mathfrak{q}} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$. Quaternions can be represented by 2×2 complex matrices:

$$\mathfrak{q} = q_0\sigma_0 + i\mathbf{q} \cdot \underline{\sigma}, \tag{2.1}$$

with $q_0 \in \mathbb{R}$, $\mathbf{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$, $\sigma_0 = \mathbb{I}_2$, the 2×2 identity matrix, and $\underline{\sigma} = (\sigma_1, -\sigma_2, \sigma_3)$, where the σ_ℓ, $\ell = 1, 2, 3$ are the usual Pauli matrices. The quaternionic imaginary units are identified as, $\mathbf{i} = \sqrt{-1}\sigma_1$, $\mathbf{j} = -\sqrt{-1}\sigma_2$, $\mathbf{k} = \sqrt{-1}\sigma_3$. Thus,

$$\mathfrak{q} = \begin{pmatrix} q_0 + iq_3 & -q_2 + iq_1 \\ q_2 + iq_1 & q_0 - iq_3 \end{pmatrix} \tag{2.2}$$

and $\overline{\mathfrak{q}} = \mathfrak{q}^\dagger$ (matrix adjoint) . Using the polar coordinates:

$$\begin{aligned} q_0 &= r\cos\theta, \\ q_1 &= r\sin\theta\sin\phi\cos\psi, \\ q_2 &= r\sin\theta\sin\phi\sin\psi, \\ q_3 &= r\sin\theta\cos\phi, \end{aligned}$$

where $(r, \phi, \theta, \psi) \in [0, \infty) \times [0, \pi]^2 \times [0, 2\pi)$, we may write

$$\mathfrak{q} = A(r)e^{i\theta\sigma(\widehat{n})}, \tag{2.3}$$

where

$$A(r) = r\sigma_0 \tag{2.4}$$

and

$$\sigma(\widehat{n}) = \begin{pmatrix} \cos\phi & \sin\phi e^{i\psi} \\ \sin\phi e^{-i\psi} & -\cos\phi \end{pmatrix}. \tag{2.5}$$

The matrices $A(r)$ and $\sigma(\widehat{n})$ satisfy the conditions,

$$A(r) = A(r)^\dagger,\ \sigma(\widehat{n})^2 = \sigma_0,\ \sigma(\widehat{n})^\dagger = \sigma(\widehat{n}) \tag{2.6}$$

and $[A(r), \sigma(\widehat{n})] = 0$. Note that a real norm on $\mathbb{H}$ is defined by

$$|\mathfrak{q}|^2 := \overline{\mathfrak{q}}\mathfrak{q} = r^2\sigma_0 = (q_0^2 + q_1^2 + q_2^2 + q_3^2).$$

Note also that for $\mathfrak{p}, \mathfrak{q} \in \mathbb{H}$, we have $\overline{\mathfrak{p}\mathfrak{q}} = \overline{\mathfrak{q}}\,\overline{\mathfrak{p}}$, $\mathfrak{p}\mathfrak{q} \neq \mathfrak{q}\mathfrak{p}$, $\mathfrak{q}\overline{\mathfrak{q}} = \overline{\mathfrak{q}}\mathfrak{q}$, and real numbers commute with quaternions. Quaternions can also be interpreted as a sum of scalar and a vector by writing

$$\mathfrak{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = (q_0, \mathbf{q});$$

where $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. Now we borrow some materials as needed here from [11]. Let

$$\mathbb{S} = \{I = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mid x_1, x_2, x_3 \in \mathbb{R},\ x_1^2 + x_2^2 + x_3^2 = 1\},$$

we call it a quaternion sphere.

Proposition 2.1 [11] *For any non-real quaternion $\mathfrak{q} \in \mathbb{H} \setminus \mathbb{R}$, there exist, and are unique, $x, y \in \mathbb{R}$ with $y > 0$, and $I_\mathfrak{q} \in \mathbb{S}$ such that $\mathfrak{q} = x + I_\mathfrak{q} y$.*

For every quaternion $I \in \mathbb{S}$, the complex line $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ passing through the origin, and containing 1 and I, is called a quaternion slice. Thereby, we can see that

$$\mathbb{H} = \bigcup_{I\in\mathbb{S}} \mathbb{C}_I \quad \text{and} \quad \bigcap_{I\in\mathbb{S}} \mathbb{C}_I = \mathbb{R} \tag{2.7}$$

One can also easily see that $\mathbb{C}_I \subset \mathbb{H}$ is commutative, while, elements from two different quaternion slices, $\mathbb{C}_I$ and $\mathbb{C}_J$ (for $I, J \in \mathbb{S}$ with $I \neq J$), do not necessarily commute.

2.2 Quaternionic Hilbert Spaces

In this subsection we introduce right quaternionic Hilbert spaces. For details we refer the reader to [1]. We also define the Hilbert space of square integrable functions on quaternions based on [6, 13, 21].

2.2.1 Right Quaternionic Hilbert Space

Let $V_{\mathbb{H}}^{R}$ be a linear vector space under right multiplication by quaternionic scalars (again $\mathbb{H}$ standing for the field of quaternions). For $f, g, h \in V_{\mathbb{H}}^{R}$ and $\mathfrak{q} \in \mathbb{H}$, the inner product

$$\langle \cdot \mid \cdot \rangle : V_{\mathbb{H}}^{R} \times V_{\mathbb{H}}^{R} \longrightarrow \mathbb{H}$$

satisfies the following properties

(i) $\overline{\langle f \mid g \rangle} = \langle g \mid f \rangle$
(ii) $\|f\|^2 = \langle f \mid f \rangle > 0$ unless $f = 0$, a real norm
(iii) $\langle f \mid g + h \rangle = \langle f \mid g \rangle + \langle f \mid h \rangle$
(iv) $\langle f \mid g\mathfrak{q} \rangle = \langle f \mid g \rangle \mathfrak{q}$
(v) $\langle f\mathfrak{q} \mid g \rangle = \overline{\mathfrak{q}} \langle f \mid g \rangle$

where $\overline{\mathfrak{q}}$ stands for the quaternionic conjugate. We assume that the space $V_{\mathbb{H}}^{R}$ is complete under the norm given above. Then, together with $\langle \cdot \mid \cdot \rangle$ this defines a right quaternionic Hilbert space, which we shall assume to be separable. Quaternionic Hilbert spaces share most of the standard properties of complex Hilbert spaces. In particular, the Cauchy-Schwarz inequality holds on quaternionic Hilbert spaces as well as the Riesz representation theorem for their duals. Thus, the Dirac bra-ket notation can be adapted to quaternionic Hilbert spaces:

$$\mid f\mathfrak{q} \rangle = \mid f \rangle \mathfrak{q}, \ \langle f\mathfrak{q} \mid = \overline{\mathfrak{q}} \langle f \mid ,$$

for a right quaternionic Hilbert space, with $|f\rangle$ denoting the vector f and $\langle f|$ its dual vector. Similarly the left quaternionic Hilbert space $V_{\mathbb{H}}^{L}$ can also be described, see for more detail [1, 17, 19]. The field of quaternions $\mathbb{H}$ itself can be turned into a left quaternionic Hilbert space by defining the inner product $\langle \mathfrak{q} \mid \mathfrak{q}' \rangle = \mathfrak{q}\mathfrak{q}'^{\dagger} = \mathfrak{q}\overline{\mathfrak{q}'}$ or into a right quaternionic Hilbert space with $\langle \mathfrak{q} \mid \mathfrak{q}' \rangle = \mathfrak{q}^{\dagger}\mathfrak{q}' = \overline{\mathfrak{q}}\mathfrak{q}'$. Further note that, due to the non-commutativity of quaternions the sum $\sum_{m=0}^{\infty} \mathfrak{p}^m \mathfrak{q}^m / m!$ cannot be written as $\exp(\mathfrak{p}\mathfrak{q})$. However, in any Hilbert space the norm convergence implies the convergence of the series and $\sum_{m=0}^{\infty} |\mathfrak{p}^m \mathfrak{q}^m / m!| \leqslant e^{|\mathfrak{p}||\mathfrak{q}|}$, therefore $\sum_{m=0}^{\infty} \mathfrak{p}^m \mathfrak{q}^m / m! = e_{*}^{\mathfrak{p}\mathfrak{q}}$ converges.

2.2.2 Quaternionic Hilbert Spaces of Square Integrable Functions

Let (X, μ) be a measure space and $\mathbb{H}$ the field of quaternions, then

$$L_{\mathbb{H}}^{2}(X, d\mu) = \left\{ f : X \to \mathbb{H} \middle| \int_X |f(x)|^2 d\mu(x) < \infty \right\}$$

is a right quaternionic Hilbert space which is denoted by $L^2_{\mathbb{H}}(X,\mu)$, with the (right) scalar product

$$\langle f \mid g\rangle = \int_X \overline{f(x)}g(x)d\mu(x), \tag{2.8}$$

where $\overline{f(x)}$ is the quaternionic conjugate of $f(x)$, and (right) scalar multiplication $f\mathfrak{a}$, $\mathfrak{a}\in\mathbb{H}$, with $(f\mathfrak{a})(\mathfrak{q}) = f(\mathfrak{q})\mathfrak{a}$ (see [13, 21] for details). Similarly, one could define a left quaternionic Hilbert space of square integrable functions.

2.3 Left Scalar Multiplications on $V_{\mathbb{H}}^R$

We shall extract the definition and some properties of left scalar multiples of vectors on $V_{\mathbb{H}}^R$ from [12] as needed for the development of the manuscript. The left scalar multiple of vectors on a right quaternionic Hilbert space is an extremely non-canonical operation associated with a choice of preferred Hilbert basis. Now the Hilbert space $V_{\mathbb{H}}^R$ has a orthonormal basis (Hilbert basis)

$$\mathcal{O} = \{\varphi_k \mid k\in N\}, \tag{2.9}$$

where N is a countable index set. The left scalar multiplication '$\cdot$' on $V_{\mathbb{H}}^R$ induced by $\mathcal{O}$ is defined as the map $\mathbb{H}\times V_{\mathbb{H}}^R \ni (\mathfrak{q},\phi)\longmapsto \mathfrak{q}\cdot\phi\in V_{\mathbb{H}}^R$ given by

$$\mathfrak{q}\cdot\phi := \sum_{k\in N}\varphi_k\mathfrak{q}\langle\varphi_k\mid\phi\rangle, \tag{2.10}$$

for all $(\mathfrak{q},\phi)\in\mathbb{H}\times V_{\mathbb{H}}^R$. Since all left multiplications are made with respect to some basis, assume that the basis $\mathcal{O}$ given by (2.9) is fixed all over the paper.

Proposition 2.2 [12] *The left product defined in (2.10) satisfies the following properties. For every $\phi,\psi\in V_{\mathbb{H}}^R$ and $\mathfrak{p},\mathfrak{q}\in\mathbb{H}$,*

(a) $\mathfrak{q}\cdot(\phi+\psi) = \mathfrak{q}\cdot\phi+\mathfrak{q}\cdot\psi$ *and* $\mathfrak{q}\cdot(\phi\mathfrak{p}) = (\mathfrak{q}\cdot\phi)\mathfrak{p}$.
(b) $\|\mathfrak{q}\cdot\phi\| = |\mathfrak{q}|\|\phi\|$.
(c) $\mathfrak{q}\cdot(\mathfrak{p}\cdot\phi) = (\mathfrak{q}\mathfrak{p}\cdot\phi)$.
(d) $\langle\overline{\mathfrak{q}}\cdot\phi\mid\psi\rangle = \langle\phi\mid\mathfrak{q}\cdot\psi\rangle$.
(e) $r\cdot\phi = \phi r$, *for all* $r\in\mathbb{R}$.
(f) $\mathfrak{q}\cdot\varphi_k = \varphi_k\mathfrak{q}$, *for all* $k\in N$.

Remark 2.3 It is immediate that $(\mathfrak{p}+\mathfrak{q})\cdot\phi = \mathfrak{p}\cdot\phi+\mathfrak{q}\cdot\phi$, for all $\mathfrak{p},\mathfrak{q}\in\mathbb{H}$ and $\phi\in V_{\mathbb{H}}^R$. Moreover, with the aid of (b) in above Proposition (2.2), we can have, if $\{\phi_n\}$ in $V_{\mathbb{H}}^R$ such that $\phi_n\longrightarrow\phi$, then $\mathfrak{q}\cdot\phi_n\longrightarrow\mathfrak{q}\cdot\phi$. Also if $\sum_n\phi_n$ is a convergent sequence in $V_{\mathbb{H}}^R$, then $\mathfrak{q}\cdot(\sum_n\phi_n) = \sum_n\mathfrak{q}\cdot\phi_n$.

Furthermore, the quaternionic scalar multiplication of $\mathbb{H}$-linear operators is also defined in [12]. For any fixed $\mathfrak{q}\in\mathbb{H}$ and a given right $\mathbb{H}$-linear operator A :

$D(A) \longrightarrow V_{\mathbb{H}}^R$, the left scalar multiplication '$\cdot$' of A is defined as a map $\mathfrak{q}\cdot A : D(A) \longrightarrow V_{\mathbb{H}}^R$ by the setting

$$(\mathfrak{q}\cdot A)\phi := \mathfrak{q}\cdot(A\phi) = \sum_{k\in N}\varphi_k\mathfrak{q}\langle\varphi_k \mid A\phi\rangle, \tag{2.11}$$

for all $\phi \in D(A)$, the domain of A. It is straightforward that $\mathfrak{q}A$ is a right $\mathbb{H}$-linear operator. If $\mathfrak{q}\cdot\phi \in D(A)$, for all $\phi \in D(A)$, one can define right scalar multiplication '$\cdot$' of the right $\mathbb{H}$-linear operator $A : D(A) \longrightarrow V_{\mathbb{H}}^R$ as a map $A\cdot\mathfrak{q} : D(A) \longrightarrow V_{\mathbb{H}}^R$ by the setting

$$(A\cdot\mathfrak{q})\phi := A(\mathfrak{q}\cdot\phi), \tag{2.12}$$

for all $\phi \in D(A)$. It is also a right $\mathbb{H}$-linear operator. One can easily obtain that, if $\mathfrak{q}\cdot\phi \in D(A)$, for all $\phi \in D(A)$ and $D(A)$ is dense in $V_{\mathbb{H}}^R$, then

$$(\mathfrak{q}\cdot A)^\dagger = A^\dagger\cdot\overline{\mathfrak{q}} \;\text{ and }\; (A\cdot\mathfrak{q})^\dagger = \overline{\mathfrak{q}}\cdot A^\dagger. \tag{2.13}$$

3 Bargmann Space of Regular and Anti-Regular Functions

The Bargmann space of left regular functions $\mathfrak{H}_r^B$ is a closed subspace of the right Hilbert space $L_{\mathbb{H}}(\mathbb{H}, d\zeta(r,\theta,\phi,\psi))$, where $d\zeta(r,\theta,\phi,\psi) = \frac{1}{4\pi}e^{-r^2}\sin\phi\, dr d\theta d\phi d\psi$. An orthonormal basis of this space is given by the monomials (which are both left and right regular)

$$\Phi_n(\mathfrak{q}) = \frac{\mathfrak{q}^n}{\sqrt{n!}};\quad n = 0, 1, 2, \cdots .$$

Similarly the vectors

$$\overline{\Phi_n(\mathfrak{q})} = \Phi_n(\overline{\mathfrak{q}}) = \frac{\overline{\mathfrak{q}}^n}{\sqrt{n!}};\quad n = 0, 1, 2, \cdots$$

form an orthonormal basis in the corresponding space of right anti-regular functions $\mathfrak{H}_{ar}^B$. There is also an associated reproducing kernel

$$K_B(\mathfrak{q},\overline{\mathfrak{p}}) = \sum_{n=0}^{\infty}\Phi_n(\mathfrak{q})\overline{\Phi_n(\mathfrak{p})} = e_\star^{\mathfrak{q}\overline{\mathfrak{p}}} \tag{3.1}$$

see [5, 19] for details.

3.1 Coherent States on Right Quaternionic Hilbert Spaces

The main content of this section is extracted from [20] as needed here. For an enhanced explanation we refer the reader to [20]. In [20] the authors have defined coherent states (CS) on $V_{\mathbb{H}}^R$ and $V_{\mathbb{H}}^L$, and also established the normalization and resolution of the identities for each of them.

On the Bargmann space $\mathfrak{H}_r^B$, the normalized canonical coherent states are

$$\eta_{\mathfrak{q}} = \frac{1}{\sqrt{K_B(\mathfrak{q},\overline{\mathfrak{q}})}}\sum_{n=0}^{\infty}\Phi_n\Phi_n(\mathfrak{q}) = e^{-\frac{|\mathfrak{q}|^2}{2}}\sum_{n=0}^{\infty}\Phi_n\frac{\mathfrak{q}^n}{n!} = e^{-\frac{|\mathfrak{q}|^2}{2}}\sum_{n=0}^{\infty}\frac{\mathfrak{q}^n}{n!}\cdot\Phi_n, \tag{3.2}$$

where we have used the fact in Proposition 2.2 (f), with a resolution of the identity

$$\int_{\mathbb{H}} |\eta_{\mathfrak{q}}\rangle\langle\eta_{\mathfrak{q}}| d\zeta(r,\theta,\phi,\psi) = I_{\mathfrak{H}_r^B}. \tag{3.3}$$

Now take the corresponding annihilation and creation operators as

$$\begin{aligned} \mathsf{a}\Phi_0 &= 0 \\ \mathsf{a}\Phi_n &= \sqrt{n}\Phi_{n-1}; \quad \forall n > 1 \\ \mathsf{a}^\dagger\Phi_n &= \sqrt{n+1}\Phi_{n+1}; \quad \forall n \geqslant 0. \end{aligned}$$

The operators can be taken as $\mathsf{a}^\dagger = \mathfrak{q}$ (multiplication by $\mathfrak{q}$) and $\mathsf{a} = \partial_s$ (left slice regular derivative), see [17, 19]. It is also not difficult to see that $(\mathsf{a}^\dagger)^\dagger = \mathsf{a}$, $[\mathsf{a}, \mathsf{a}^\dagger] = I_{\mathfrak{H}_r^B}$ and $\mathsf{a}\eta_{\mathfrak{q}} = \mathfrak{q}\cdot\eta_{\mathfrak{q}}$. Further $N = \mathsf{a}^\dagger\mathsf{a}$ serves as the number operator (see also [18]).

First of all, as in the complex quantum mechanics, all the operators considered here are unbounded operators. However, the operators act as $\mathfrak{H}_r^B \ni |\phi\rangle \mapsto |\psi\rangle \in \mathfrak{H}_r^B$, that is, the domain and the range of the operators are dense subsets of $\mathfrak{H}_r^B$. Furthermore, the Hilbert space, $\mathfrak{H}_r^B$, can be taken as a space right-spanned by the regular functions $\{\frac{\mathfrak{q}^m}{m!} \mid m \in \mathbb{N}\}$ or anti-regular functions $\{\frac{\overline{\mathfrak{q}}^m}{m!} \mid m \in \mathbb{N}\}$ over $\mathbb{H}$ (counterparts of holomorphic and anti-holomorphic functions). In this respect, the operators considered here do not have any domain problems as for the operators in the complex quantum mechanics. Therefore, we can use the operator tools of complex quantum mechanics, in particular, the Baker-Campbell-Hausdorff formula (for a complex argument along these lines see chapter 14 in [8]).

The following Proposition demonstrate commutativity between quaternions and the right linear operators a and $\mathsf{a}^\dagger$. Further, it plays an important role. This proposition is not necessarily true for general quaternionic linear operators.

Proposition 3.1 [18] *For each* $\mathfrak{q} \in \mathbb{H}$, *we have* $\mathfrak{q}\cdot\mathsf{a} = \mathsf{a}\cdot\mathfrak{q}$ *and* $\mathfrak{q}\cdot\mathsf{a}^\dagger = \mathsf{a}^\dagger\cdot\mathfrak{q}$.

3.2 The Right Quaternionic Displacement Operator

On a right quaternionic Hilbert space with a right multiplication we cannot have a displacement operator as a representation for the representation space $\mathfrak{H}_r^B$. This fact has been indicated twice in the literature, in [2] while studying quaternionic Perelomov type CS and in [20] when the authors studied the quaternionic canonical CS. However, in [18], we have shown that if we consider a right quaternionic Hilbert space with a left multiplication on it, see (2.11), we can have a displacement operator as a representation for the representation space $\mathfrak{H}_r^B$ with all the desired properties. In fact, the map $\mathfrak{q} \mapsto \mathfrak{D}(\mathfrak{q})$ is a projective representation of the additive Abelian group $\mathbb{H}$, since the composition of operators $\mathfrak{D}(\mathfrak{q})$ and $\mathfrak{D}(\mathfrak{p})$ produce another displacement operator with a phase factor [18]. We shall extract some materials from [18] as needed here.

Proposition 3.2 [18] *The right quaternionic displacement operator* $\mathfrak{D}(\mathfrak{q}) = e^{\mathfrak{q}\cdot\mathsf{a}^\dagger - \overline{\mathfrak{q}}\cdot\mathsf{a}}$ *is a unitary, square integrable and irreducible representation of the representation space* $\mathfrak{H}_r^B$.

The following proposition discusses two versions for the displacement operator, the complex version is commonly used in complex quantum mechanics without hesitation, namely normal and anti-normal orderings.

Proposition 3.3 [18] *The displacement operator* $\mathfrak{D}(\mathfrak{q})$ *satisfies*

(i) *the normal ordering property:* $\mathfrak{D}(\mathfrak{q}) = e^{-\frac{|\mathfrak{q}|^2}{2}} e^{\mathfrak{q}\cdot\mathsf{a}^\dagger} e^{-\overline{\mathfrak{q}}\cdot\mathsf{a}}$,

(ii) *the anti-normal ordering property:* $\mathfrak{D}(\mathfrak{q}) = e^{\frac{|\mathfrak{q}|^2}{2}} e^{-\overline{\mathfrak{q}}\cdot\mathsf{a}} e^{\mathfrak{q}\cdot\mathsf{a}^\dagger}$.

Furthermore, the coherent state $\eta_{\mathfrak{q}}$ *is generated from the ground state* Φ_0 *by the displacement operator* $\mathfrak{D}(\mathfrak{q})$,

$$\eta_{\mathfrak{q}} = \mathfrak{D}(\mathfrak{q})\Phi_0. \tag{3.4}$$

Proposition 3.4 [18] *The displacement operator* $\mathfrak{D}(\mathfrak{q})$ *satisfies the following properties*

$$(i)\ \ \mathfrak{D}(\mathfrak{q})^\dagger \mathsf{a} \mathfrak{D}(\mathfrak{q}) = \mathsf{a} + \mathfrak{q} \qquad (ii)\ \ \mathfrak{D}(\mathfrak{q})^\dagger \mathsf{a}^\dagger \mathfrak{D}(\mathfrak{q}) = \mathsf{a}^\dagger + \overline{\mathfrak{q}}.$$

3.3 The Right Quaternionic Squeeze Operator

Same reason as for the displacement operator, with a right multiplication on a right quaternionic Hilbert space the squeezed operator cannot be unitary. However, it becomes unitary with a left multiplication on a right quaternionic Hilbert space.

Lemma 3.5 *The operator* $A = \mathfrak{p}\cdot(\mathsf{a}^\dagger)^2 - \overline{\mathfrak{p}}\cdot\mathsf{a}^2$ *is anti-hermitian.*

Proof Consider

$$\begin{aligned} A^\dagger &= (\mathfrak{p}\cdot(\mathsf{a}^\dagger)^2 - \overline{\mathfrak{p}}\cdot\mathsf{a}^2)^\dagger \\ &= ((\mathsf{a}^\dagger)^2)^\dagger\cdot\overline{\mathfrak{p}} - (\mathsf{a}^2)^\dagger\cdot\mathfrak{p} \quad \text{by } 2.13 \\ &= \mathsf{a}^2\cdot\overline{\mathfrak{p}} - (\mathsf{a}^\dagger)^2\cdot\mathfrak{p} \\ &= \overline{\mathfrak{p}}\cdot\mathsf{a}^2 - \mathfrak{p}\cdot(\mathsf{a}^\dagger)^2 \quad \text{by Prop.3.1} \\ &= -A. \end{aligned}$$

□

Let $A^\dagger = -A = B$, then A and B commute and both commute with the commutator $[A,B]$. Further $e^{-\frac{1}{2}[A,B]} = 1$, therefore by the Baker-Campbell-Hausdorff formula,

$$e^A e^B e^{-\frac{1}{2}[A,B]} = e^{A+B}$$

we have, for the operator

$$S(\mathfrak{p}) = e^{\frac{1}{2}(\mathfrak{p}\cdot(\mathsf{a}^\dagger)^2 - \overline{\mathfrak{p}}\cdot\mathsf{a}^2)},$$

$$S(\mathfrak{p})S(\mathfrak{p})^\dagger = e^{\frac{1}{2}A} e^{\frac{1}{2}A^\dagger} = e^{\frac{1}{2}(A-A)} = I_{\mathfrak{H}_r^B}$$

and similarly $S(\mathfrak{p})^\dagger S(\mathfrak{p}) = I_{\mathfrak{H}_r^B}$. That is the operator $S(\mathfrak{p})$ is unitary and we call this operator the *quaternionic squeeze operator*. Further

$$S(\mathfrak{p})^\dagger = e^{-\frac{1}{2}A} = S(-\mathfrak{p}).$$

If we take

$$K_+ = \frac{1}{2}(\mathsf{a}^\dagger)^2, \quad K_- = \frac{1}{2}\mathsf{a}^2, \quad \text{and} \quad K_0 = \frac{1}{2}(\mathsf{a}^\dagger\mathsf{a} + \frac{1}{2}I_{\mathfrak{H}_r^B}),$$

Then they satisfy the commutation rules

$$[K_0, K_+] = K_+, \quad [K_0, K_-] = -K_-, \quad \text{and} \quad [K_+, K_-] = -2K_0.$$

That is, K_+, K_- and K_0 are the generators of the $su(1,1)$ algebra and they satisfy the $su(1,1)$ commutation rules. In terms of these operators the squeeze operator $S(\mathfrak{p})$ can be written as

$$S(\mathfrak{p}) = e^{\mathfrak{p}\cdot K_+ - \overline{\mathfrak{p}}\cdot K_-}. \tag{3.5}$$

3.4 Some Quaternionic Lie Algebras

In the complex quantum mechanics the generators $\{N, \mathsf{a}^2, (\mathsf{a}^\dagger)^2\}$ spans the Lie algebra $su(1,1)$ and this algebra is involved in the construction of pure squeezed states. The generators $\{\mathsf{a}, \mathsf{a}^\dagger, I, N, \mathsf{a}^2, (\mathsf{a}^\dagger)^2\}$ spans a six dimensional algebra $\mathfrak{h}_6$, which is involved in the construction of the generalized squeezed states [10]. In the following we generalize it to quaternions.

Let $\tau \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and define

$$\mathfrak{h}_6^{(\tau)} = \text{linear span over } \mathbb{C}_\tau \, \{\mathbb{I}_{\mathfrak{H}_r^B}, \mathsf{a}, \mathsf{a}^\dagger, N = \mathsf{a}^\dagger\mathsf{a}, \mathsf{a}^2, (\mathsf{a}^\dagger)^2\};$$

where $\mathbb{C}_\tau = \{x = x_1 + \tau x_2 \mid x_1, x_2 \in \mathbb{R}\}$. Then the Proposition 2.2 guarantees, together with the Remark 2.3, that $\mathfrak{h}_6^{(\tau)}$ is a vector space over $\mathbb{C}_\tau$ under the left multiplication '$\cdot$' which is defined in (2.11). Define

$$[\cdot,\cdot]_\tau : \mathfrak{h}_6^{(\tau)} \times \mathfrak{h}_6^{(\tau)} \longrightarrow \mathfrak{h}_6^{(\tau)} \quad \text{by} \quad [\mathcal{A}, \mathcal{B}]_\tau = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}, \quad \text{for all } \mathcal{A}, \mathcal{B} \in \mathfrak{h}_6^{(\tau)}. \tag{3.6}$$

One can easily see that, for any $\mathcal{A}, \mathcal{B} \in \mathfrak{h}_6^{(\tau)}$, $[\mathcal{A}, \mathcal{B}]_\tau \in \mathfrak{h}_6^{(\tau)}$, using the facts that

$$[\mathsf{a}, \mathsf{a}^\dagger]_\tau = \mathbb{I}_{\mathfrak{H}_r^B}, \qquad [\mathsf{a}, N]_\tau = \mathsf{a}, \qquad [\mathsf{a}^\dagger, N]_\tau = -\mathsf{a}^\dagger, \qquad [\mathsf{a}^2, (\mathsf{a}^\dagger)^2]_\tau = -2(2N + \mathbb{I}_{\mathfrak{H}_r^B}),$$

$$[\mathsf{a}^2, \mathsf{a}^\dagger]_\tau = 2\mathsf{a}, \quad [(\mathsf{a}^\dagger)^2, \mathsf{a}]_\tau = -2\mathsf{a}^\dagger, \quad [\mathsf{a}^2, N]_\tau = 2\mathsf{a}^2, \qquad [(\mathsf{a}^\dagger)^2, N]_\tau = -2(\mathsf{a}^\dagger)^2.$$

with the aid of Proposition 3.1. Hence $\mathfrak{h}_6^{(\tau)}$ is a Lie algebra with the Lie bracket $[\cdot,\cdot]_\tau$.

One can easily check that the subset (but it is a linear space itself over $\mathbb{R}$) of $\mathfrak{h}_6^{(\tau)}$,

$$\text{linear span over } \mathbb{R} \, \{\mathbb{I}_{\mathfrak{H}_r^B}, \mathsf{a}, \mathsf{a}^\dagger, N = \mathsf{a}^\dagger\mathsf{a}, \mathsf{a}^2, (\mathsf{a}^\dagger)^2\}$$

forms a Lie algebra with the Lie bracket $[\mathcal{A}, \mathcal{B}]_1 = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$, for all elements $\mathcal{A}, \mathcal{B}$ in this linear space (it is a restriction map of $[\cdot,\cdot]_\tau$). Furthermore, we have another subset

$$\mathfrak{h}_{12}^{(\tau)} = \text{linear span over } \mathbb{R} \, \{\mathbb{I}_{\mathfrak{H}_r^B}, N, \mathsf{a}, \mathsf{a}^\dagger, \mathsf{a}^2, (\mathsf{a}^\dagger)^2, \tau\cdot\mathbb{I}_{\mathfrak{H}_r^B}, \tau\cdot N, \tau\cdot\mathsf{a}, \tau\cdot\mathsf{a}^\dagger, \tau\cdot\mathsf{a}^2, \tau\cdot(\mathsf{a}^\dagger)^2\}$$

which is a linear space over $\mathbb{R}$, and forms a Lie algebra with the Lie bracket $[\cdot,\cdot]_\tau$. Moreover, An arbitrary element $\mathcal{A} \in \mathfrak{h}_{12}^{(\tau)}$ takes the form

$$\begin{aligned}\mathcal{A} &= a_1 \cdot \mathbb{I}_{\mathfrak{H}_r^B} + a_2 \cdot N + a_3 \cdot \mathsf{a} + a_4 \cdot \mathsf{a}^\dagger + a_5 \cdot \mathsf{a}^2 + a_6 \cdot (\mathsf{a}^\dagger)^2 + a_\tau^{(1)}\tau \cdot \mathsf{a}\\ &\quad + a_\tau^{(2)}\tau \cdot \mathsf{a}^\dagger + a_\tau^{(3)}\tau \cdot \mathsf{a}^2 + a_\tau^{(4)}\tau \cdot (\mathsf{a}^\dagger)^2; \end{aligned} \tag{3.7}$$

where $a_l, a_\tau^{(m)} \in \mathbb{R}$, for all $l = 1, 2, \cdots, 6$, $m = 1, 2, \cdots, 4$. It can be simply expressed as

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_\tau; \tag{3.8}$$

where

$$\mathcal{A}_1 = a_1 \cdot \mathbb{I}_{\mathfrak{H}_r^B} + a_2 \cdot N + a_3 \cdot \mathsf{a} + a_4 \cdot \mathsf{a}^\dagger + a_5 \cdot \mathsf{a}^2 + a_6 \cdot (\mathsf{a}^\dagger)^2, \tag{3.9}$$

$$\mathcal{A}_\tau = a_\tau^{(1)}\tau \cdot \mathbb{I}_{\mathfrak{H}_r^B} + a_\tau^{(2)}\tau \cdot N + a_\tau^{(3)}\tau \cdot \mathsf{a} + a_\tau^{(4)}\tau \cdot \mathsf{a}^\dagger + a_\tau^{(5)}\tau \cdot \mathsf{a}^2 + a_\tau^{(6)}\tau \cdot (\mathsf{a}^\dagger)^2. \tag{3.10}$$

Only for notational convenience, we shall write $\mathfrak{q} = q_0 + q_\mathbf{i}\mathbf{i} + q_\mathbf{j}\mathbf{j} + q_\mathbf{k}\mathbf{k}$ with $q_0, q_\mathbf{i}, q_\mathbf{j}, q_\mathbf{k} \in \mathbb{R}$ as $\mathfrak{q} = q_0 + \sum_{\tau=\mathbf{i},\mathbf{j},\mathbf{k}} q_\tau \tau$. Let

$$\mathfrak{h}_{24} = \text{linear span over } \mathbb{R}\,\{\tau \cdot \mathbb{I}_{\mathfrak{H}_r^B}, \tau \cdot N, \tau \cdot \mathsf{a}, \tau \cdot \mathsf{a}^\dagger, \tau \cdot \mathsf{a}^2, \tau \cdot (\mathsf{a}^\dagger)^2 \mid \tau = 1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}.$$

Then $\mathfrak{h}_{24}$ is a vector space over $\mathbb{R}$, and it contains $\mathfrak{h}_{12}^{(\tau)}$. Define the map $[\cdot,\cdot] : \mathfrak{h}_{24} \times \mathfrak{h}_{24} \longrightarrow \mathfrak{h}_{24}$, by the setting (using the expression (3.8))

$$[\mathcal{A}, \mathcal{B}] := [\mathcal{A}_1, \mathcal{B}_1]_1 + \sum_{\tau=\mathbf{i},\mathbf{j},\mathbf{k}} [\mathcal{A}_1, \mathcal{B}_\tau]_\tau + \sum_{\tau=\mathbf{i},\mathbf{j},\mathbf{k}} [\mathcal{A}_\tau, \mathcal{B}_1 + \mathcal{B}_\tau]_\tau, \quad \text{for all} \quad \mathcal{A}, \mathcal{B} \in \mathfrak{h}_{24}. \tag{3.11}$$

Alternatively it can be written as

$$[\mathcal{A}, \mathcal{B}] = \sum_{\tau=\mathbf{i},\mathbf{j},\mathbf{k}} \left(\frac{1}{3}[\mathcal{A}_1, \mathcal{B}_1]_\tau + [\mathcal{A}_1, \mathcal{B}_\tau]_\tau + [\mathcal{A}_\tau, \mathcal{B}_1 + \mathcal{B}_\tau]_\tau\right), \quad \text{for all} \quad \mathcal{A}, \mathcal{B} \in \mathfrak{h}_{24}. \tag{3.12}$$

Since $[\cdot,\cdot]_\tau$ is a Lie bracket, it follows that $\mathfrak{h}_{24}$ is a Lie algebra with the Lie bracket $[\cdot,\cdot]$. The following Proposition can be proved using this Lie bracket $[\cdot,\cdot]$.

Proposition 3.6 *Let* $\mathfrak{p} = |\mathfrak{p}|e^{i\theta\sigma(\hat{n})}$ *and* $N = \mathsf{a}^\dagger\mathsf{a}$*, the number operator, then the squeeze operator* $S(\mathfrak{p})$ *satisfies the following relations*

$$\begin{aligned} (i)\ S(\mathfrak{p})^\dagger \mathsf{a} S(\mathfrak{p}) &= (\cosh|\mathfrak{p}|)\mathsf{a} + \left(e^{i\theta\sigma(\hat{n})}\sinh|\mathfrak{p}|\right) \cdot \mathsf{a}^\dagger.\\ (ii)\ S(\mathfrak{p})^\dagger \mathsf{a}^\dagger S(\mathfrak{p}) &= (\cosh|\mathfrak{p}|)\mathsf{a}^\dagger + \left(e^{-i\theta\sigma(\hat{n})}\sinh|\mathfrak{p}|\right) \cdot \mathsf{a}.\\ (iii)\ S(\mathfrak{p})^\dagger N S(\mathfrak{p}) &= (\cosh^2|\mathfrak{p}|)\mathsf{a}^\dagger\mathsf{a} + \left(e^{-i\theta\sigma(\hat{n})}\sinh|\mathfrak{p}|\cosh|\mathfrak{p}|\right) \cdot \mathsf{a}^2\\ &\quad + \left(e^{i\theta\sigma(\hat{n})}\sinh|\mathfrak{p}|\cosh|\mathfrak{p}|\right) \cdot (\mathsf{a}^\dagger)^2 + (\sinh^2|\mathfrak{p}|)\mathsf{a}\mathsf{a}^\dagger. \end{aligned}$$

Proof With $A = \frac{1}{2}(\mathfrak{p} \cdot (\mathsf{a}^\dagger)^2 - \overline{\mathfrak{p}} \cdot \mathsf{a}^2)$ and the commutation rule $[\mathsf{a}, \mathsf{a}^\dagger] = I_{\mathfrak{H}_r^B}$ we can calculate

$$\begin{aligned}
&[-A, \mathsf{a}] = \mathfrak{p} \cdot \mathsf{a}^\dagger \\
&[-A, [-A, \mathsf{a}]] = |\mathfrak{p}|^2 \mathsf{a} \\
&[-A, [-A, [-A, \mathsf{a}]]] = |\mathfrak{p}|^2 \mathfrak{p} \cdot \mathsf{a}^\dagger \\
&[-A, [-A, [-A, [-A, \mathsf{a}]]]] = |\mathfrak{p}|^4 \mathsf{a} \\
&[-A, [-A, [-A, [-A, [-A, \mathsf{a}]]]]] = |\mathfrak{p}|^4 \mathfrak{p} \cdot \mathsf{a}^\dagger \\
&\cdots\cdots\cdots
\end{aligned}$$

Therefore, by using the identity $e^C B e^{-C} = B + [C, B] + \frac{1}{2!}[C, [C, B]] + \cdots$ we have

$$\begin{aligned}
S(\mathfrak{p})^\dagger \mathsf{a} S(\mathfrak{p}) &= \mathsf{a} + \mathfrak{p} \cdot \mathsf{a}^\dagger + \frac{1}{2!}|\mathfrak{p}|^2 \mathsf{a} + \frac{1}{3!}|\mathfrak{p}|^2 \mathfrak{p} \cdot \mathsf{a}^\dagger + \frac{1}{4!}|\mathfrak{p}|^4 \mathsf{a} + \frac{1}{5!}|\mathfrak{p}|^4 \mathfrak{p} \cdot \mathsf{a}^\dagger + \cdots \\
&= (\mathsf{a} + \frac{1}{2!}|\mathfrak{p}|^2 \mathsf{a} + \frac{1}{4!}|\mathfrak{p}|^4 \mathsf{a} + \cdots) \\
&\quad + e^{i\theta\sigma(\hat{n})} \cdot (|\mathfrak{p}| \mathsf{a}^\dagger + \frac{1}{3!}|\mathfrak{p}|^3 \mathsf{a}^\dagger + \frac{1}{5!}|\mathfrak{p}|^5 \mathsf{a}^\dagger + \cdots) \\
&= \left(\sum_{n=0}^{\infty} \frac{|\mathfrak{p}|^{2n}}{(2n)!} \right) \mathsf{a} + e^{i\theta\sigma(\hat{n})} \left(\sum_{n=0}^{\infty} \frac{|\mathfrak{p}|^{2n+1}}{(2n+1)!} \right) \cdot \mathsf{a}^\dagger \\
&= (\cosh |\mathfrak{p}|) \mathsf{a} + \left(e^{i\theta\sigma(\hat{n})} \sinh |\mathfrak{p}| \right) \cdot \mathsf{a}^\dagger .
\end{aligned}$$

The second relation is the hermitian conjugate of the first one. The third relation can be obtained by writing

$$S(\mathfrak{p})^\dagger N S(\mathfrak{p}) = S(\mathfrak{p}) \mathsf{a}^\dagger \mathsf{a} S(\mathfrak{p}) = S(\mathfrak{p}) \mathsf{a} S(\mathfrak{p}) S(\mathfrak{p})^\dagger \mathsf{a} S(\mathfrak{p})$$

and then multiplying the first and the second relations. □

3.5 Right Quaternionic Quadrature Operators

We introduce the quadrature operators analogous to the complex quadrature operators with a left multiplication on a right quaternionic Hilbert space.

$$X = \frac{1}{2}(\mathsf{a} + \mathsf{a}^\dagger) \quad \text{and} \quad Y = -\frac{\mathbf{i}}{2} \cdot (\mathsf{a} - \mathsf{a}^\dagger), \tag{3.13}$$

where the quaternion unit $\mathbf{i}$ in Y can be replaced by $\mathbf{j}$, $\mathbf{k}$ or any $I \in \mathbb{S}$ (see [18]).

Proposition 3.7 *The operators X and Y are self-adjoint and* $[X, Y] = \frac{\mathbf{i}}{2} \cdot I_{\mathfrak{H}_r^B}$.

Proof The self-adjointness of the operators follows prom the Prop. 3.1 (see [18]). Now it is straight forward to compute, with the aid of Prop. 3.1,

$$XY = -\frac{1}{4}(\mathsf{a} + \mathsf{a}^\dagger)(\mathbf{i} \cdot (\mathsf{a} - \mathsf{a}^\dagger) = -\frac{1}{4}\mathbf{i} \cdot (\mathsf{a}^2 - \mathsf{a}\mathsf{a}^\dagger + \mathsf{a}^\dagger \mathsf{a} - (\mathsf{a}^\dagger)^2)$$

and

$$YX = -\frac{1}{4}\mathbf{i} \cdot (\mathsf{a} - \mathsf{a}^\dagger)(\mathsf{a} + \mathsf{a}^\dagger) = -\frac{1}{4}\mathbf{i} \cdot (\mathsf{a}^2 + \mathsf{a}\mathsf{a}^\dagger - \mathsf{a}^\dagger\mathsf{a} - (\mathsf{a}^\dagger)^2).$$

Thus

$$[X, Y] = XY - YX = -\frac{1}{2}\mathbf{i} \cdot (-\mathsf{a}\mathsf{a}^\dagger + \mathsf{a}^\dagger\mathsf{a}) = \frac{1}{2}\mathbf{i} \cdot [\mathsf{a}, \mathsf{a}^\dagger] = \frac{1}{2}\mathbf{i} \cdot I_{\mathfrak{H}_r^B}.$$

□

Definition 3.8 [10] Let A and B be quantum observables with commutator $[A, B] = \mathbf{i} \cdot C$. When variances are calculated in a generic state, one obtains from Cauchy-Schwarz inequality $(\Delta A)(\Delta B) \geqslant \frac{1}{2}|\langle C\rangle|$. A state will be called squeezed with respect to the pair (A, B) if $(\Delta A)^2$ (or $(\Delta B)^2$) $< \frac{1}{2}|\langle C\rangle|$. A state is called ideally squeezed if the equality $(\Delta A)(\Delta B) = \frac{1}{2}|\langle C\rangle|$ is reached together with $(\Delta A)^2$ (or $(\Delta B)^2$) $< \frac{1}{2}|\langle C\rangle|$.

We adapt the same definition for quaternionic squeezed states.

3.6 Right Quaternionic Pure Squeezed States

A pure squeezed state is produced by the sole action of the unitary operator $S(\mathfrak{p})$ on the vacuum state. That is, $\eta_{\mathfrak{p}} = S(\mathfrak{p})\Phi_0$ is the pure squeezed state. Even though a series expression is not necessary for the computations of the expectation values, we provide the following for the sake of completeness. Let $C = \frac{1}{2}\mathfrak{p} \cdot (\mathsf{a}^\dagger)^2$ and $D = \frac{1}{2}\overline{\mathfrak{p}} \cdot \mathsf{a}^2$. Then it can be computed that

$$\begin{aligned}
[C, D]\Phi_n &= -\frac{1}{4}\left(\mathfrak{p} \cdot (\mathsf{a}^\dagger)^2\, \overline{\mathfrak{p}} \cdot \mathsf{a}^2 - \overline{\mathfrak{p}} \cdot \mathsf{a}^2\, \mathfrak{p} \cdot (\mathsf{a}^\dagger)^2\right)\Phi_n \\
&= \frac{1}{4}|\mathfrak{p}|^2\left((\mathsf{a}^\dagger)^2\mathsf{a}^2 - \mathsf{a}^2(\mathsf{a}^\dagger)^2\right)\Phi_n \\
&= -\frac{1}{2}|\mathfrak{p}|^2(2n+1)\Phi_n.
\end{aligned}$$

That is $[C, D] = -\frac{1}{2}|\mathfrak{p}|^2(2n+1)I_{\mathfrak{H}_r^B}$. Further, similarly, we can obtain

$$[C, [C, D]]\Phi_n = 0 \quad \text{and} \quad [D, [C, D]]\Phi_n = 0.$$

That is $[C, [C, D]] = 0$ and $[D, [C, D]] = 0$. Therefore from the BCH formula we have

$$S(\mathfrak{p}) = e^{C-D} = e^{\frac{1}{2}[C,D]}e^{C}e^{-D}.$$

Now

$$\begin{aligned}
S(\mathfrak{p})\Phi_0 &= e^{\frac{1}{2}[C,D]}e^{C}e^{-D}\Phi_0\\
&= e^{\frac{1}{2}[C,D]}e^{C}\Phi_0\\
&= e^{\frac{1}{2}[C,D]}\sum_{n=0}^{\infty}\frac{(\mathfrak{p}\cdot(\mathsf{a}^{\dagger})^2)^n}{2^n n!}\Phi_0\\
&= e^{\frac{1}{2}[C,D]}\sum_{n=0}^{\infty}\frac{\mathfrak{p}^n\sqrt{(2n)!}}{2^n n!}\cdot\Phi_{2n}.
\end{aligned}$$

Further,

$$\begin{aligned}
[C,D]\Phi_{2n} &= (\frac{1}{2}\mathfrak{p}\cdot(\mathsf{a}^{\dagger})^2\frac{1}{2}\overline{\mathfrak{p}}\cdot\mathsf{a}^2-\frac{1}{2}\overline{\mathfrak{p}}\cdot\mathsf{a}^2\frac{1}{2}\mathfrak{p}(\mathsf{a}^{\dagger})^2)\Phi_{2n}\\
&= \frac{1}{4}|\mathfrak{p}|^2((\mathsf{a}^{\dagger})^2\mathsf{a}^2-\mathsf{a}^2(\mathsf{a}^{\dagger})^2)\Phi_{2n}\\
&= \frac{1}{4}|\mathfrak{p}|^2(-8n-2)\Phi_{2n}
\end{aligned}$$

Therefore

$$e^{\frac{1}{2}[C,D]}\Phi_{2n}=e^{-\frac{1}{8}|\mathfrak{p}|^2(8n+2)}\Phi_{2n}=e^{-n|\mathfrak{p}|^2}e^{-\frac{1}{4}|\mathfrak{p}|^2}\Phi_{2n}.$$

Thus

$$S(\mathfrak{p})\Phi_0=\eta_{\mathfrak{p}}=e^{-\frac{1}{4}|\mathfrak{p}|^2}\sum_{n=0}^{\infty}e^{-n|\mathfrak{p}|^2}\frac{\mathfrak{p}^n\sqrt{(2n)!}}{2^n n!}\cdot\Phi_{2n}. \tag{3.14}$$

Since $S(\mathfrak{p})$ is a unitary operator, by construction we have

$$\langle\eta_{\mathfrak{p}}|\eta_{\mathfrak{p}}\rangle=\langle S(\mathfrak{p})\Phi_0|S(\mathfrak{p})\Phi_0\rangle=\langle\Phi_0|\Phi_0\rangle=1. \tag{3.15}$$

The states $\eta_{\mathfrak{p}}$ are normalized. Since the pure squeezed state $\eta_{\mathfrak{p}}$ only possess the even numbered basis vector, $\{\Phi_{2n} \mid n=0,1,2,\cdots\}$ a resolution of the identity cannot hold on $\mathfrak{H}_r^B$. However, if we form a space right spanned by $\{\Phi_{2n} \mid n=0,1,2,\cdots\}$ over the quaternions it may be possible to find a resolution of the identity for that space. However, such an attempt is not necessary and even in the complex case, as far as we know, it does not exist in the literature.

3.6.1 Expectation Values and the Variances

For a normalized state η the expectation value of an operator F is $\langle F\rangle = \langle\eta|F|\eta\rangle$. First let us see the expectation values of a and $\mathsf{a}^\dagger$ using Proposition 3.6.

$$\begin{aligned}
\langle\mathsf{a}\rangle &= \langle\eta_{\mathfrak{p}}|\mathsf{a}|\eta_{\mathfrak{p}}\rangle = \langle S(\mathfrak{p})\Phi_0|\mathsf{a}|S(\mathfrak{p})\Phi_0\rangle\\
&= \langle\Phi_0|S(\mathfrak{p})^\dagger\mathsf{a}S(\mathfrak{p})\Phi_0\rangle\\
&= \langle\Phi_0|(\cosh|\mathfrak{p}|)\mathsf{a} + \left(e^{i\theta\sigma(\hat{n})}\sinh|\mathfrak{p}|\right)\cdot\mathsf{a}^\dagger\Phi_0\rangle\\
&= (\cosh|\mathfrak{p}|)\langle\Phi_0|\mathsf{a}\Phi_0\rangle + \sinh|\mathfrak{p}|\langle\Phi_0|\left(e^{i\theta\sigma(\hat{n})}\right)\cdot\mathsf{a}^\dagger\Phi_0\rangle\\
&= 0 + \sinh|\mathfrak{p}|\langle\Phi_0|\left(e^{i\theta\sigma(\hat{n})}\right)\cdot\Phi_1\rangle\\
&= \sinh|\mathfrak{p}|\langle\Phi_0|\Phi_1 e^{i\theta\sigma(\hat{n})}\rangle \quad \text{as } \Phi_1 \text{ is a basis vector, see Prop.2.2 (f)}\\
&= \sinh|\mathfrak{p}|\langle\Phi_0|\Phi_1\rangle e^{i\theta\sigma(\hat{n})} = 0.
\end{aligned}$$

Similarly we get

$$\langle\mathsf{a}^\dagger\rangle = \langle\eta_{\mathfrak{p}}|\mathsf{a}^\dagger|\eta_{\mathfrak{p}}\rangle = 0.$$

Hence we get

$$\langle X\rangle = \langle\eta_{\mathfrak{p}}|X|\eta_{\mathfrak{p}}\rangle = 0 \quad\text{and}\quad \langle Y\rangle = \langle\eta_{\mathfrak{p}}|Y|\eta_{\mathfrak{p}}\rangle = 0. \tag{3.16}$$

Since

$$\begin{aligned}
&S(\mathfrak{p})^\dagger\mathsf{a}S(\mathfrak{p})S(\mathfrak{p})^\dagger\mathsf{a}^\dagger S(\mathfrak{p}) = S(\mathfrak{p})^\dagger\mathsf{a}\mathsf{a}^\dagger S(\mathfrak{p})\\
&= \cosh^2|\mathfrak{p}|\mathsf{a}\mathsf{a}^\dagger + e^{-i\theta\sigma(\hat{n})}\cosh|\mathfrak{p}|\sinh|\mathfrak{p}|\cdot\mathsf{a}^2\\
&\quad + e^{i\theta\sigma(\hat{n})}\cosh|\mathfrak{p}|\sinh|\mathfrak{p}|\cdot(\mathsf{a}^\dagger)^2 + \sinh^2|\mathfrak{p}|\mathsf{a}^\dagger\mathsf{a}
\end{aligned}$$

and similarly

$$\begin{aligned}
&S(\mathfrak{p})^\dagger\mathsf{a}^\dagger\mathsf{a}S(\mathfrak{p})\\
&= \cosh^2|\mathfrak{p}|\mathsf{a}^\dagger\mathsf{a} + e^{-i\theta\sigma(\hat{n})}\cosh|\mathfrak{p}|\sinh|\mathfrak{p}|\cdot\mathsf{a}^2\\
&\quad + e^{i\theta\sigma(\hat{n})}\cosh|\mathfrak{p}|\sinh|\mathfrak{p}|\cdot(\mathsf{a}^\dagger)^2 + \sinh^2|\mathfrak{p}|\mathsf{a}\mathsf{a}^\dagger,
\end{aligned}$$

$$\begin{aligned}
S(\mathfrak{p})^\dagger\mathsf{a}^2S(\mathfrak{p}) &= \cosh^2|\mathfrak{p}|\mathsf{a}^2 + e^{i\theta\sigma(\hat{n})}\cosh|\mathfrak{p}|\sinh|\mathfrak{p}|\cdot(\mathsf{a}\mathsf{a}^\dagger + \mathsf{a}^\dagger\mathsf{a})\\
&\quad + e^{2i\theta\sigma(\hat{n})}\sinh^2|\mathfrak{p}|\cdot(\mathsf{a}^\dagger)^2,\\
S(\mathfrak{p})^\dagger(\mathsf{a}^\dagger)^2S(\mathfrak{p}) &= \cosh^2|\mathfrak{p}|(\mathsf{a}^\dagger)^2 + e^{-i\theta\sigma(\hat{n})}\cosh|\mathfrak{p}|\sinh|\mathfrak{p}|\cdot(\mathsf{a}\mathsf{a}^\dagger + \mathsf{a}^\dagger\mathsf{a})\\
&\quad + e^{-2i\theta\sigma(\hat{n})}\sinh^2|\mathfrak{p}|\cdot\mathsf{a}^2.
\end{aligned}$$

Using the above relations we readily obtain

$$\begin{aligned}
\langle\mathsf{a}\mathsf{a}^\dagger\rangle &= \langle\eta_{\mathfrak{p}}|\mathsf{a}\mathsf{a}^\dagger|\eta_{\mathfrak{p}}\rangle = \cosh^2|\mathfrak{p}|,\\
\langle\mathsf{a}^\dagger\mathsf{a}\rangle &= \langle\eta_{\mathfrak{p}}|\mathsf{a}^\dagger\mathsf{a}|\eta_{\mathfrak{p}}\rangle = \sinh^2|\mathfrak{p}|,\\
\langle\mathsf{a}^2\rangle &= \langle\eta_{\mathfrak{p}}|\mathsf{a}^2|\eta_{\mathfrak{p}}\rangle = \cosh|\mathfrak{p}|\sinh|\mathfrak{p}|e^{i\theta\sigma(\hat{n})},\\
\langle(\mathsf{a}^\dagger)^2\rangle &= \langle\eta_{\mathfrak{p}}|(\mathsf{a}^\dagger)^2|\eta_{\mathfrak{p}}\rangle = \cosh|\mathfrak{p}|\sinh|\mathfrak{p}|e^{-i\theta\sigma(\hat{n})}.
\end{aligned}$$

Since $X^2=\frac{1}{4}((\mathsf{a}^\dagger)^2+\mathsf{a}\mathsf{a}^\dagger+\mathsf{a}^\dagger\mathsf{a}+\mathsf{a}^2)$ and $Y^2=\frac{1}{4}(\mathsf{a}\mathsf{a}^\dagger+\mathsf{a}^\dagger\mathsf{a}-\mathsf{a}^2-(\mathsf{a}^\dagger)^2)$ we have

$$\begin{aligned}\langle\eta_{\mathfrak{p}}|X^2|\eta_{\mathfrak{p}}\rangle &= \frac{1}{4}\left\{(\cosh^2|\mathfrak{p}|+\sinh^2|\mathfrak{p}|)\mathbb{I}_2+\cosh|\mathfrak{p}|\sinh|\mathfrak{p}|\left(e^{i\theta\sigma(\hat{n})}+e^{-i\theta\sigma(\hat{n})}\right)\right\}\\ &= \frac{1}{4}\left\{\cosh(2|\mathfrak{p}|)\mathbb{I}_2+\sinh(2|\mathfrak{p}|)\cos(\theta\sigma(\hat{n}))\right\},\\ \langle\eta_{\mathfrak{p}}|Y^2|\eta_{\mathfrak{p}}\rangle &= \frac{1}{4}\left\{\cosh(2|\mathfrak{p}|)\mathbb{I}_2-\sinh(2|\mathfrak{p}|)\cos(\theta\sigma(\hat{n}))\right\}.\end{aligned}$$

Since $\langle\Delta X\rangle^2=\langle\eta_{\mathfrak{p}}|X^2|\eta_{\mathfrak{p}}\rangle-\langle\eta_{\mathfrak{p}}|X|\eta_{\mathfrak{p}}\rangle^2$ and $\langle\eta_{\mathfrak{p}}|X|\eta_{\mathfrak{p}}\rangle=0$ we have

$$\begin{aligned}\langle\Delta X\rangle^2 &= \frac{1}{4}\left\{\cosh(2|\mathfrak{p}|)\mathbb{I}_2+\sinh(2|\mathfrak{p}|)\cos(\theta\sigma(\hat{n}))\right\},\\ \langle\Delta Y\rangle^2 &= \frac{1}{4}\left\{\cosh(2|\mathfrak{p}|)\mathbb{I}_2-\sinh(2|\mathfrak{p}|)\cos(\theta\sigma(\hat{n}))\right\}.\end{aligned}$$

Hence

$$\begin{aligned}\langle\Delta X\rangle^2\langle\Delta Y\rangle^2 &= \frac{1}{16}\left\{\cosh^2(2|\mathfrak{p}|)\mathbb{I}_2-\sinh^2(2|\mathfrak{p}|)\cos^2(\theta\sigma(\hat{n}))\right\}\\ &= \frac{1}{16}\left\{\cosh^2(2|\mathfrak{p}|)\mathbb{I}_2-\sinh^2(2|\mathfrak{p}|)(1-\sin^2(\theta\sigma(\hat{n})))\right\}\\ &= \frac{1}{16}\left\{\mathbb{I}_2+\sinh^2(2|\mathfrak{p}|)\sin^2(\theta\sigma(\hat{n}))\right\}.\end{aligned}$$

An exact analogue of the complex case. Since we are in the quaternions, it appears as a 2×2 matrix. Further in the complex case, the product of the variances depends on r and θ (when $z=re^{i\theta}$). In the quaternion case it depends on all four parameters r,θ,ϕ and ψ. Let us write

$$U+\mathbf{i}\cdot V=e^{-\frac{i}{2}\theta\sigma(\hat{n})}\cdot(X+\mathbf{i}\cdot Y)=e^{-\frac{i}{2}\theta\sigma(\hat{n})}\cdot\mathsf{a}. \tag{3.17}$$

Then using Proposition 3.1 we can write

$$\begin{aligned}S(\mathfrak{p})^\dagger(U+\mathbf{i}\cdot V)S(\mathfrak{p}) &= e^{-\frac{i}{2}\theta\sigma(\hat{n})}\cdot S(\mathfrak{p})^\dagger\mathsf{a}S(\mathfrak{p})\\ &= e^{-\frac{i}{2}\theta\sigma(\hat{n})}\cdot(\cosh|\mathfrak{p}|\mathsf{a}+e^{i\theta\sigma(\hat{n})}\sinh|\mathfrak{p}|\mathsf{a}^\dagger)\\ &= \frac{e^{|\mathfrak{p}|}+e^{-|\mathfrak{p}|}}{2}e^{-\frac{i}{2}\theta\sigma(\hat{n})}\cdot\mathsf{a}+\frac{e^{|\mathfrak{p}|}-e^{-|\mathfrak{p}|}}{2}e^{\frac{i}{2}\theta\sigma(\hat{n})}\cdot\mathsf{a}^\dagger\\ &= \frac{1}{2}(e^{-\frac{i}{2}\theta\sigma(\hat{n})}\cdot\mathsf{a}+e^{\frac{i}{2}\theta\sigma(\hat{n})}\cdot\mathsf{a}^\dagger)e^{|\mathfrak{p}|}\\ &\quad+\frac{1}{2}(e^{-\frac{i}{2}\theta\sigma(\hat{n})}\cdot\mathsf{a}-e^{\frac{i}{2}\theta\sigma(\hat{n})}\cdot\mathsf{a}^\dagger)e^{-|\mathfrak{p}|}\\ &= Ue^{|\mathfrak{p}|}+\mathbf{i}\cdot Ve^{-|\mathfrak{p}|},\end{aligned}$$

with

$$\begin{aligned}U &= \frac{1}{2}(e^{-\frac{i}{2}\theta\sigma(\hat{n})}\cdot\mathsf{a}+e^{\frac{i}{2}\theta\sigma(\hat{n})}\cdot\mathsf{a}^\dagger)e^{|\mathfrak{p}|}\quad\text{and}\\ V &= -\frac{i}{2}\cdot(e^{-\frac{i}{2}\theta\sigma(\hat{n})}\cdot\mathsf{a}-e^{\frac{i}{2}\theta\sigma(\hat{n})}\cdot\mathsf{a}^\dagger)e^{|\mathfrak{p}|}.\end{aligned}$$

Since

$$U^2 = \frac{1}{4}(e^{-i\theta\sigma(\hat{n})}\cdot \mathsf{a}^2 + \mathsf{a}\mathsf{a}^\dagger + \mathsf{a}^\dagger\mathsf{a} + e^{i\theta\sigma(\hat{n})}\cdot(\mathsf{a}^\dagger)^2) \quad \text{and}$$
$$V^2 = -\frac{1}{4}(e^{-i\theta\sigma(\hat{n})}\cdot \mathsf{a}^2 - \mathsf{a}\mathsf{a}^\dagger - \mathsf{a}^\dagger\mathsf{a} + e^{i\theta\sigma(\hat{n})}\cdot(\mathsf{a}^\dagger)^2),$$

it is straight forward that $\langle\eta_{\mathfrak{p}}|U|\eta_{\mathfrak{p}}\rangle = 0$, $\langle\eta_{\mathfrak{p}}|V|\eta_{\mathfrak{p}}\rangle = 0$,

$$\begin{aligned}\langle\eta_{\mathfrak{p}}|U^2|\eta_{\mathfrak{p}}\rangle &= \frac{1}{4}(\cosh|\mathfrak{p}|\sinh|\mathfrak{p}| + \cosh^2|\mathfrak{p}| + \sinh^2|\mathfrak{p}| + \cosh|\mathfrak{p}|\sinh|\mathfrak{p}|)\mathbb{I}_2\\ &= \frac{1}{4}(\cosh|\mathfrak{p}| + \sinh|\mathfrak{p}|)^2\mathbb{I}_2 \quad \text{and}\\ \langle\eta_{\mathfrak{p}}|V^2|\eta_{\mathfrak{p}}\rangle &= -\frac{1}{4}(\cosh|\mathfrak{p}|\sinh|\mathfrak{p}| - \cosh^2|\mathfrak{p}| - \sinh^2|\mathfrak{p}| + \cosh|\mathfrak{p}|\sinh|\mathfrak{p}|)\mathbb{I}_2\\ &= \frac{1}{4}(\cosh|\mathfrak{p}| - \sinh|\mathfrak{p}|)^2\mathbb{I}_2.\end{aligned}$$

Hence

$$\langle\Delta U\rangle^2\langle\Delta V\rangle^2 = \frac{1}{16}(\cosh^2|\mathfrak{p}| - \sinh^2|\mathfrak{p}|)^2\mathbb{I}_2 = \frac{1}{16}\mathbb{I}_2 \tag{3.18}$$

and therefore

$$\langle\Delta U\rangle\langle\Delta V\rangle = \frac{1}{4}\mathbb{I}_2, \tag{3.19}$$

while $\langle\Delta U\rangle \neq \langle\Delta V\rangle$, an exact analogue of the complex case [10]. Hence, the class of ideally squeezed states with respect to the operators U, V contains the set of quaternionic pure squeezed states.

Using the relation (iii) in Proposition 3.6 we obtain the mean photon number

$$\langle N\rangle = \langle\eta_{\mathfrak{p}}|N|\eta_{\mathfrak{p}}\rangle = \langle\Phi_0|S(\mathfrak{p})^\dagger N S(\mathfrak{p})\Phi_0\rangle = \sinh^2|\mathfrak{p}|\mathbb{I}_2. \tag{3.20}$$

Also using

$$\langle\eta_{\mathfrak{p}}|N^2|\eta_{\mathfrak{p}}\rangle = \langle\Phi_0|S(\mathfrak{p})^\dagger N S(\mathfrak{p})S(\mathfrak{p})^\dagger N S(\mathfrak{p})\Phi_0\rangle$$

we get

$$\begin{aligned}\langle N^2\rangle &= \langle\Phi_0|S(\mathfrak{p})^\dagger N S(\mathfrak{p})S(\mathfrak{p})^\dagger N S(\mathfrak{p})\Phi_0\rangle\\ &= (\sinh^4|\mathfrak{p}| + 2\sinh^2|\mathfrak{p}|\cosh^2|\mathfrak{p}|)\mathbb{I}_2\\ &= 3\sinh^4|\mathfrak{p}| + 2\sinh^2|\mathfrak{p}|\mathbb{I}_2.\end{aligned}$$

Hence the variance is

$$\langle\Delta N\rangle^2 = \langle N^2\rangle - \langle N\rangle^2 = 2\sinh^2|\mathfrak{p}|(1 + \sinh^2|\mathfrak{p}|)\mathbb{I}_2. \tag{3.21}$$

The photon number variance is also described by Mandel's Q-parameter. The Mandel parameter is [10, 14]

$$\begin{aligned}Q_M &= \frac{\langle\Delta N\rangle^2}{\langle N\rangle} - 1\\ &= \frac{2\sinh^2|\mathfrak{p}|(1 + \sinh^2|\mathfrak{p}|)}{\sinh^2|\mathfrak{p}|}\mathbb{I}_2 - \mathbb{I}_2\\ &= (1 + 2\sinh^2|\mathfrak{p}|)\mathbb{I}_2 = 2\langle N\rangle + \mathbb{I}_2.\end{aligned}$$

Since $Q_M > 0$ (as a positive definite matrix) the photon number probability distribution is super-Poissonian. ($Q_M = 0$ Poissonian and $Q_M < 0$ sub-Poissonian).

3.6.2 The Pure Squeezed States with Anti-Normal Ordering of $S(\mathfrak{p})$

Even though in order to compute the expectation values and variances the relations in Proposition 3.6 are enough, let us give an expression for the pure squeezed states with the anti-normal ordering of the operator $S(\mathfrak{p})$.

$$\begin{aligned}
\eta_{\mathfrak{p}}^{a} &= S(\mathfrak{p})\Phi_0\\
&= e^{\frac{1}{2}[C,D]}e^{-D}e^{C}\Phi_0\\
&= e^{\frac{1}{2}[C,D]}e^{-D}\sum_{m=0}^{\infty}\frac{(\mathfrak{p}\cdot(\mathsf{a}^{\dagger})^2)^m}{2^m m!}\Phi_0\\
&= e^{\frac{1}{2}[C,D]}e^{-D}\sum_{m=0}^{\infty}\frac{\mathfrak{p}^m\sqrt{(2m)!}}{2^m m!}\cdot\Phi_{2m}\\
&= e^{\frac{1}{2}[C,D]}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathfrak{p}^m\sqrt{(2m)!}}{2^m m!}\frac{\overline{\mathfrak{p}}^n\mathsf{a}^{2n}}{2^n n!}\cdot\Phi_{2m}\\
&= e^{\frac{1}{2}[C,D]}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{\mathfrak{p}^m\sqrt{(2m)!}}{2^m m!}\sqrt{\frac{(2m)!}{(2m-2n)!}}\frac{\overline{\mathfrak{p}}^n}{2^n n!}\cdot\Phi_{2m-2n}\\
&= e^{\frac{1}{2}[C,D]}\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{\mathfrak{p}^m\overline{\mathfrak{p}}^n(2m)!}{2^{m+n}m!n!\sqrt{(2m-2n)!}}\cdot\Phi_{2m-2n}\quad\text{assuming } m>n\\
&= e^{\frac{1}{2}[C,D]}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\frac{\mathfrak{p}^{n+s}\overline{\mathfrak{p}}^n(2n+2s)!}{4^n2^s(n+s)!n!\sqrt{(2s)!}}\cdot\Phi_{2s}\quad\text{taking } m-n=s\\
&= e^{-\frac{1}{4}|\mathfrak{p}|^2}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\frac{\mathfrak{p}^{n+s}\overline{\mathfrak{p}}^n(2n+2s)!}{4^n2^s(n+s)!n!\sqrt{(2s)!}}e^{-s|\mathfrak{p}|^2}\cdot\Phi_{2s}.
\end{aligned}$$

3.7 Right Quaternionic Squeezed States

In view of Prop. 2.2(f), for a basis vector $\mathfrak{q}\cdot\Phi_n = \Phi_n\mathfrak{q}$, therefore we write the canonical CS as

$$\eta_{\mathfrak{q}} = \mathfrak{D}(\mathfrak{q})\Phi_0 = e^{-|\mathfrak{q}|^2/2}\sum_{n=0}^{\infty}\Phi_n\frac{\mathfrak{q}^n}{\sqrt{n!}}.$$

Let $S(\mathfrak{p})\Phi_n = \Phi_n^{\mathfrak{p}}$, where the set $\{\Phi_n \mid n = 0,1,2,\cdots\}$ is the basis of the Fock space of regular Bargmann space $\mathfrak{H}_r^B$. Since $S(\mathfrak{p})$ is a unitary operator, the set $\{\Phi_n^{\mathfrak{p}} \mid n = 0,1,2,\cdots\}$ is also form an orthonormal basis for $\mathfrak{H}_r^B$. That is

$$\langle\Phi_m^{\mathfrak{p}}|\Phi_n^{\mathfrak{p}}\rangle = \delta_{mn}. \tag{3.22}$$

Now the squeezed states are

$$\eta_{\mathfrak{q}}^{\mathfrak{p}} = S(\mathfrak{p})\mathfrak{D}(\mathfrak{q})\Phi_0 = S(\mathfrak{p})\eta_{\mathfrak{q}} = e^{-|\mathfrak{q}|^2/2}\sum_{n=0}^{\infty}\Phi_n^{\mathfrak{p}}\frac{\mathfrak{q}^n}{\sqrt{n!}}. \tag{3.23}$$

Since the canonical CS are normalized, that is $\langle\eta_{\mathfrak{q}}|\eta_{\mathfrak{q}}\rangle = 1$, and the squeeze operator $S(\mathfrak{p})$ is unitary, we have

$$\langle\eta_{\mathfrak{q}}^{\mathfrak{p}}|\eta_{\mathfrak{q}}^{\mathfrak{p}}\rangle = \langle S(\mathfrak{p})\eta_{\mathfrak{q}}|S(\mathfrak{p})\eta_{\mathfrak{q}}\rangle = \langle\eta_{\mathfrak{q}}|\eta_{\mathfrak{q}}\rangle = 1.$$

That is, the squeezed states are normalized. The dual vector of $|S(\mathfrak{p})\eta_{\mathfrak{q}}\rangle$ is $\langle\eta_{\mathfrak{q}}S(\mathfrak{p})^{\dagger}|$. Therefore, from the resolution of the identity of the canonical CS,

$$\int_{\mathbb{H}}|\eta_{\mathfrak{q}}\rangle\langle\eta_{\mathfrak{q}}|d\zeta(r,\theta,\phi,\psi) = I_{\mathfrak{H}_r^B}$$

we get

$$\int_{\mathbb{H}}|S(\mathfrak{p})\eta_{\mathfrak{q}}\rangle\langle\eta_{\mathfrak{q}}S(\mathfrak{p})^{\dagger}|d\zeta(r,\theta,\phi,\psi) = S(\mathfrak{p})I_{\mathfrak{H}_r^B}S(\mathfrak{p})^{\dagger} = I_{\mathfrak{H}_r^B}.$$

That is the squeezed states satisfy the resolution of the identity,

$$\int_{\mathbb{H}}|\eta_{\mathfrak{q}}^{\mathfrak{p}}\rangle\langle\eta_{\mathfrak{q}}^{\mathfrak{p}}|d\zeta(r,\theta,\phi,\psi) = I_{\mathfrak{H}_r^B}.$$

Remark 3.9 Since the operators $\mathfrak{D}(\mathfrak{p})$ and $S(\mathfrak{q})$ are unitary operators the two photon states $\mathfrak{D}(\mathfrak{p})S(\mathfrak{q})\Phi_0$ are normalized but, for the same reason as for the pure squeezed states, these states cannot hold a resolution of the identity in the space $\mathfrak{H}_r^B$. Further, technically, a series expansion for the states $S(\mathfrak{p})\mathfrak{D}(\mathfrak{q})\Phi_0$ and $\mathfrak{D}(\mathfrak{p})S(\mathfrak{q})\Phi_0$ can be obtained. However, it is rather complicated and not necessary (even in the complex case).

In the complex case, combining the results in the Propositions 3.4 and 3.6 (corresponding complex case) one can obtain a relation for the operators, for $z, \xi \in \mathbb{C}$,

$$S(\xi)^{\dagger}\mathfrak{D}(z)^{\dagger}\mathsf{a}\mathfrak{D}(\xi)S(z) \quad \text{and} \quad S(\xi)^{\dagger}\mathfrak{D}(z)^{\dagger}\mathsf{a}^{\dagger}\mathfrak{D}(\xi)S(z)$$

or for the operators

$$\mathfrak{D}(\xi)^{\dagger}S(z)^{\dagger}\mathsf{a}S(z)\mathfrak{D}(\xi) \quad \text{and} \quad \mathfrak{D}(\xi)^{\dagger}S(z)^{\dagger}\mathsf{a}^{\dagger}S(z)\mathfrak{D}(\xi)$$

and use them to compute the expectation values and variances of all the required operators. Since quaternions do not commute such relations cannot be obtained for quaternions. For example, if we combine the Propositions 3.4 and 3.6, when $\mathfrak{p} = |\mathfrak{p}|e^{i\theta\sigma(\hat{n})}$ let $I_{\mathfrak{p}} = e^{i\theta\sigma(\hat{n})}$,

$$\begin{aligned}
\mathfrak{D}(\mathfrak{q})^{\dagger}S(\mathfrak{p})^{\dagger}\mathsf{a}S(\mathfrak{p})\mathfrak{D}(\mathfrak{q}) &= \mathfrak{D}(\mathfrak{q})^{\dagger}\left[(\cosh|\mathfrak{p}|)\mathsf{a} + I_{\mathfrak{p}}\sinh|\mathfrak{p}|\cdot\mathsf{a}^{\dagger}\right]\mathfrak{D}(\mathfrak{q})\\
&= \cosh|\mathfrak{p}|\mathfrak{D}(\mathfrak{q})^{\dagger}\mathsf{a}\mathfrak{D}(\mathfrak{q}) + \sinh|\mathfrak{p}|\mathfrak{D}(\mathfrak{q})^{\dagger}I_{\mathfrak{p}}\cdot\mathsf{a}^{\dagger}\mathfrak{D}(\mathfrak{q}).
\end{aligned}$$

Since $\mathfrak{D}(\mathfrak{q})^{\dagger}I_{\mathfrak{p}}\cdot\mathsf{a}^{\dagger}\mathfrak{D}(\mathfrak{q}) \neq I_{\mathfrak{p}}\cdot\mathfrak{D}(\mathfrak{q})^{\dagger}\mathsf{a}^{\dagger}\mathfrak{D}(\mathfrak{q})$, the above expression cannot be computed. In fact, there is no know technique in quaternion analysis to get a closed form for the expression $\mathfrak{D}(\mathfrak{q})^{\dagger} I_{\mathfrak{p}} \cdot \mathsf{a}^{\dagger}\mathfrak{D}(\mathfrak{q})$. In this regard, even though we have established normalized squeezed states $S(\mathfrak{p})\mathfrak{D}(\mathfrak{q})\Phi_0$ with a resolution of the identity, the corresponding expectation values and variances cannot be obtained in a usable form. Since elements in a quaternion slice commute, if we consider squeezed states in a quaternion slice then the computations can carry forward. From the slice-wise analysis we may able to get to the whole set of quaternions $\mathbb{H}$ through direct integrals. For such an analysis with quaternionic canonical coherent states we refer to [16].

4 Squeezed States on a Quaternion Slice

Let $\mathbb{C}_I$ be a quaternion slice. Since elements in $\mathbb{C}_I$ commute we can obtain the following relations for squeezed coherent states and two photon coherent states, and obtain the related expectation values. The states $\mathfrak{D}(\mathfrak{q})S(\mathfrak{p})\Phi_0$ are called the two photon coherent states [15, 22]. On the other hand the states $S(\mathfrak{p})\mathfrak{D}(\mathfrak{q})\Phi_0$ are called the squeezed coherent states [15] pp. 207. We shall demonstrate it briefly in this section.

4.1 Squeezed States

Let $\mathfrak{p}, \mathfrak{q} \in \mathbb{C}_I$, then we can write

$$\begin{aligned}
\mathfrak{p} &= |\mathfrak{p}|e^{I\theta_{\mathfrak{p}}} = |\mathfrak{p}|I_{\mathfrak{p}} = |\mathfrak{p}|(\cos\theta_{\mathfrak{p}} + I\sin\theta_{\mathfrak{p}}) \quad \text{and} \\
\mathfrak{q} &= |\mathfrak{q}|e^{I\theta_{\mathfrak{q}}} = |\mathfrak{q}|I_{\mathfrak{q}} = |\mathfrak{q}|(\cos\theta_{\mathfrak{q}} + I\sin\theta_{\mathfrak{q}}).
\end{aligned}$$

The normalized squeezed states are $\eta_{\mathfrak{q}}^{\mathfrak{p}} = S(\mathfrak{p})\mathfrak{D}(\mathfrak{q})\Phi_0$. With these notations we obtain the following.

Proposition 4.1 *The operators $S(\mathfrak{p})$ and $\mathfrak{D}(\mathfrak{q})$ satisfy the following relations.*

$$\begin{aligned}
\mathfrak{D}(\mathfrak{q})^{\dagger}S(\mathfrak{p})^{\dagger}\mathsf{a}S(\mathfrak{p})\mathfrak{D}(\mathfrak{q}) &= \cosh|\mathfrak{p}|\mathsf{a}\mathbb{I}_2 + I_{\mathfrak{p}}\sinh|\mathfrak{p}|\cdot\mathsf{a}^{\dagger} + \cosh|\mathfrak{p}|\mathfrak{q}\mathbb{I}_2 \\
&\quad + I_{\mathfrak{p}}\sinh|\mathfrak{p}|\overline{\mathfrak{q}} \\
\mathfrak{D}(\mathfrak{q})^{\dagger}S(\mathfrak{p})^{\dagger}\mathsf{a}^{\dagger}S(\mathfrak{p})\mathfrak{D}(\mathfrak{q}) &= \cosh|\mathfrak{p}|\mathsf{a}^{\dagger}\mathbb{I}_2 + \overline{I}_{\mathfrak{p}}\sinh|\mathfrak{p}|\cdot\mathsf{a} + \cosh|\mathfrak{p}|\overline{\mathfrak{q}}\mathbb{I}_2 \\
&\quad + \overline{I}_{\mathfrak{p}}\sinh|\mathfrak{p}|\mathfrak{q}, \\
\mathfrak{D}(\mathfrak{q})^{\dagger}S(\mathfrak{p})^{\dagger}NS(\mathfrak{p})\mathfrak{D}(\mathfrak{q}) &= \cosh^2|\mathfrak{p}|(N + \mathfrak{q}\cdot\mathsf{a}^{\dagger} + \overline{\mathfrak{q}}\cdot\mathsf{a} + |\mathfrak{q}|^2) \\
&\quad + \frac{1}{2}\overline{I}_{\mathfrak{p}}\sinh(2|\mathfrak{p}|)\cdot(\mathsf{a}^2 + 2\mathfrak{q}\cdot\mathsf{a} + \mathfrak{q}^2) \\
&\quad + \frac{1}{2}I_{\mathfrak{p}}\sinh(2|\mathfrak{p}|)\cdot((\mathsf{a}^{\dagger})^2 + 2\overline{\mathfrak{q}}\cdot\mathsf{a}^{\dagger} + \overline{\mathfrak{q}}^2) \\
&\quad + \sinh^2|\mathfrak{p}|(\mathsf{a}\mathsf{a}^{\dagger} + \overline{\mathfrak{q}}\cdot\mathsf{a} + \mathfrak{q}\cdot\mathsf{a}^{\dagger} + |\mathfrak{q}|^2).
\end{aligned}$$

Proof Proof is straight forward from the results of the Propositions 3.6 and 3.4. □

For a normalized squeezed state and an operator F we denote the expectation value as $\langle F\rangle_{\mathfrak{pq}} = \langle \eta^{\mathfrak{p}}_{\mathfrak{q}}|F|\eta^{\mathfrak{p}}_{\mathfrak{q}}\rangle$. The following expectation values can be calculated.

$$\begin{aligned}
\langle \mathsf{a}\rangle_{\mathfrak{pq}} &= \cosh|\mathfrak{p}|\mathfrak{q} + I_{\mathfrak{p}}\sinh|\mathfrak{p}|\overline{\mathfrak{q}}\\
\langle \mathsf{a}^{\dagger}\rangle_{\mathfrak{pq}} &= \cosh|\mathfrak{p}|\overline{\mathfrak{q}} + \overline{I}_{\mathfrak{p}}\sinh|\mathfrak{p}|\mathfrak{q}\\
\langle X\rangle_{\mathfrak{pq}} &= |\mathfrak{q}|\left[\cosh|\mathfrak{p}|\cos\theta_{\mathfrak{q}} + \sinh|\mathfrak{p}|\cos(\theta_{\mathfrak{p}}-\theta_{\mathfrak{q}})\right]\\
\langle Y\rangle_{\mathfrak{pq}} &= |\mathfrak{q}|\left[\cosh|\mathfrak{p}|\sin\theta_{\mathfrak{q}} + \sinh|\mathfrak{p}|\sin(\theta_{\mathfrak{p}}-\theta_{\mathfrak{q}})\right]\\
\langle \mathsf{a}\mathsf{a}^{\dagger}\rangle_{\mathfrak{pq}} &= \cosh^2|\mathfrak{p}| + \cosh(2|\mathfrak{p}|)|\mathfrak{q}|^2 + |\mathfrak{q}|^2\sinh(2|\mathfrak{p}|)\cos(2\theta_{\mathfrak{q}}-\theta_{\mathfrak{p}})\\
\langle \mathsf{a}^{\dagger}\mathsf{a}\rangle_{\mathfrak{pq}} &= \sinh^2|\mathfrak{p}| + \cosh(2|\mathfrak{p}|)|\mathfrak{q}|^2 + |\mathfrak{q}|^2\sinh(2|\mathfrak{p}|)\cos(2\theta_{\mathfrak{q}}-\theta_{\mathfrak{p}})\\
\langle \mathsf{a}^2\rangle_{\mathfrak{pq}} &= \frac{1}{2}I_{\mathfrak{p}}\sinh(2|\mathfrak{p}|)(1+2|\mathfrak{q}|^2) + \cosh^2|\mathfrak{p}|\mathfrak{q}^2 + I_{\mathfrak{p}}^2\sinh^2|\mathfrak{p}|\overline{\mathfrak{q}}^2\\
\langle (\mathsf{a}^{\dagger})^2\rangle_{\mathfrak{pq}} &= \frac{1}{2}\overline{I}_{\mathfrak{p}}\sinh(2|\mathfrak{p}|)(1+2|\mathfrak{q}|^2) + \cosh^2|\mathfrak{p}|\overline{\mathfrak{q}}^2 + \overline{I}_{\mathfrak{p}}^2\sinh^2|\mathfrak{p}|\mathfrak{q}^2.
\end{aligned}$$

Using the above expectations we can readily obtain the following.

$$\begin{aligned}
\langle X^2\rangle &= \frac{1}{2}\left[\left(\cosh(2|\mathfrak{p}|) + 2|\mathfrak{q}|^2\cosh(2|\mathfrak{p}|) + 2|\mathfrak{q}|^2\sinh(2|\mathfrak{p}|)\cos(2\theta_{\mathfrak{q}}-\theta_{\mathfrak{p}})\right)\right]\\
&+ \frac{1}{2}\left[\left(\cos\theta_{\mathfrak{p}}\sinh(2|\mathfrak{p}|)(1+2|\mathfrak{q}|^2) + 2|\mathfrak{q}|^2\cosh^2|\mathfrak{p}|\cos(2\theta_{\mathfrak{q}}) + 2|\mathfrak{q}|^2\sinh^2|\mathfrak{p}|\cos(2\theta_{\mathfrak{p}}-2\theta_{\mathfrak{q}})\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
\langle Y^2\rangle &= \frac{1}{2}\left[\left(\cosh(2|\mathfrak{p}|) + 2|\mathfrak{q}|^2\cosh(2|\mathfrak{p}|) + 2|\mathfrak{q}|^2\sinh(2|\mathfrak{p}|)\cos(2\theta_{\mathfrak{q}}-\theta_{\mathfrak{p}})\right)\right]\\
&- \frac{1}{2}\left[\left(\cos\theta_{\mathfrak{p}}\sinh(2|\mathfrak{p}|)(1+2|\mathfrak{q}|^2) + 2|\mathfrak{q}|^2\cosh^2|\mathfrak{p}|\cos(2\theta_{\mathfrak{q}}) + 2|\mathfrak{q}|^2\sinh^2|\mathfrak{p}|\cos(2\theta_{\mathfrak{p}}-2\theta_{\mathfrak{q}})\right)\right].
\end{aligned}$$

Using these expectation values the variances of X and Y can be obtained.

4.2 Two Photon Coherent States

The two photon coherent states are defined as $\eta^{\mathfrak{q}}_{\mathfrak{p}} = \mathfrak{D}(\mathfrak{q})S(\mathfrak{p})\Phi_0$ [10, 15]. We briefly provide some formulas for these states. Once again we are in a quaternion slice $\mathbb{C}_I$ and $\mathfrak{p}$ and $\mathfrak{q}$ are as in the previous section.

Proposition 4.2

$$\begin{aligned}
S^{\dagger}(\mathfrak{p})\mathfrak{D}(\mathfrak{q})^{\dagger}\mathsf{a}\mathfrak{D}(\mathfrak{q})S(\mathfrak{p}) &= \cosh|\mathfrak{p}|\,\mathsf{a} + I_{\mathfrak{p}}\sinh\mathfrak{p}\,\mathsf{a}^{\dagger} + \mathfrak{q}\\
S^{\dagger}(\mathfrak{p})\mathfrak{D}(\mathfrak{q})^{\dagger}\mathsf{a}^{\dagger}\mathfrak{D}(\mathfrak{q})S(\mathfrak{p}) &= \cosh|\mathfrak{p}|\,\mathsf{a}^{\dagger} + \overline{I}_{\mathfrak{p}}\sinh\mathfrak{p}\,\mathsf{a} + \overline{\mathfrak{q}}\\
S^{\dagger}(\mathfrak{p})\mathfrak{D}(\mathfrak{q})^{\dagger}\mathsf{a}^{\dagger}\mathsf{a}\mathfrak{D}(\mathfrak{q})S(\mathfrak{p}) &= \cosh^2|\mathfrak{p}|\,\mathsf{a}^{\dagger}\mathsf{a} + \frac{1}{2}I_{\mathfrak{p}}\sinh(2|\mathfrak{p}|)\,(\mathsf{a}^{\dagger})^2 + \mathfrak{q}\cosh|\mathfrak{p}|\,\mathsf{a}^{\dagger}\\
&+ \frac{1}{2}\overline{I}_{\mathfrak{p}}\sinh(2|\mathfrak{p}|)\,\mathsf{a}^2 + \sinh^2|\mathfrak{p}|\,\mathsf{a}\mathsf{a}^{\dagger} + \overline{I}_{\mathfrak{p}}\mathfrak{q}\sinh|\mathfrak{p}|\,\mathsf{a}\\
&+ \overline{\mathfrak{q}}\cosh|\mathfrak{p}|\,\mathsf{a} + I_{\mathfrak{p}}\overline{\mathfrak{q}}\sinh|\mathfrak{p}|\,\mathsf{a}^{\dagger} + |\mathfrak{q}|^2.
\end{aligned}$$

Proof Proof is straight forward from Propositions 3.6 and 3.4. □

With the aid of the above proposition we can easily calculate the following.

$$\begin{aligned}
\langle \mathsf{a}\rangle_{\mathfrak{qp}} &= \mathfrak{q} \\
\langle \mathsf{a}^{\dagger}\rangle_{\mathfrak{qp}} &= \overline{\mathfrak{q}} \\
\langle N\rangle_{\mathfrak{qp}} &= \sinh^2|\mathfrak{p}| + |\mathfrak{q}|^2 \\
\langle X\rangle_{\mathfrak{qp}} &= |\mathfrak{q}|\cos\theta_{\mathfrak{q}} \\
\langle Y\rangle_{\mathfrak{qp}} &= |\mathfrak{q}|\sin\theta_{\mathfrak{q}} \\
\langle \mathsf{a}^2\rangle_{\mathfrak{qp}} &= \frac{1}{2}I_{\mathfrak{p}}\sinh(2|\mathfrak{p}|) + \mathfrak{q}^2 \\
\langle (\mathsf{a}^{\dagger})^2\rangle_{\mathfrak{qp}} &= \frac{1}{2}\overline{I}_{\mathfrak{p}}\sinh(2|\mathfrak{p}|) + \overline{\mathfrak{q}}^2 \\
\langle \mathsf{a}\mathsf{a}^{\dagger}\rangle_{\mathfrak{qp}} &= \cosh^2|\mathfrak{p}| + |\mathfrak{q}|^2.
\end{aligned}$$

Hence, using the above, we can calculate the following.

$$\begin{aligned}
\langle X^2\rangle_{\mathfrak{qp}} &= \frac{1}{4}\left\{\cosh(2|\mathfrak{p}|) + 2|\mathfrak{q}|^2 + \sinh(2|\mathfrak{p}|)\cos\theta_{\mathfrak{p}} + 2|\mathfrak{q}|^2\cos(2\theta_{\mathfrak{q}})\right\}, \\
\langle Y^2\rangle_{\mathfrak{qp}} &= \frac{1}{4}\left\{\cosh(2|\mathfrak{p}|) + 2|\mathfrak{q}|^2 - \sinh(2|\mathfrak{p}|)\cos\theta_{\mathfrak{p}} - 2|\mathfrak{q}|^2\cos(2\theta_{\mathfrak{q}})\right\}
\end{aligned}$$

and

$$\begin{aligned}
\langle N^2\rangle_{\mathfrak{qp}} &= \frac{1}{2}\sinh^2(2|\mathfrak{p}|) + \sinh^4|\mathfrak{p}| + 2|\mathfrak{q}|^2\sinh^2|\mathfrak{p}| + |\mathfrak{q}|^2\cosh(2|\mathfrak{p}|) \\
&\quad + |\mathfrak{q}|^2\sinh(2|\mathfrak{p}|) + |\mathfrak{q}|^4.
\end{aligned}$$

Further

$$\begin{aligned}
\langle \Delta N\rangle_{\mathfrak{qp}}^2 &= \frac{1}{2}\sinh^2(2|\mathfrak{p}|) + |\mathfrak{q}|^2\cosh(2|\mathfrak{p}|) + |\mathfrak{q}|^2\sinh(2|\mathfrak{p}|) \\
&= \frac{1}{2}\sinh^2(2|\mathfrak{p}|) + |\mathfrak{q}|^2 e^{2|\mathfrak{p}|}.
\end{aligned}$$

For a normalized state $\eta_{\mathfrak{p}}^{\mathfrak{q}}$, in terms of the quadrature operator X, the signal-to-noise ratio and the Mandel parameter are, respectively, defined as (see [10])

$$SNR = \frac{\langle X\rangle_{\mathfrak{qp}}^2}{\langle \Delta X\rangle_{\mathfrak{qp}}^2} \quad \text{and} \quad Q_M = \frac{\langle \Delta N\rangle_{\mathfrak{qp}}}{\langle N\rangle_{\mathfrak{qp}}} - 1.$$

Using the above expectation values one can easily obtain these quantities.

Proposition 4.3 *The operator $S(\mathfrak{p})$ satisfies the disentanglement formula*

$$S(\mathfrak{p}) = e^{\mathfrak{p}\cdot K_+ - \bar{\mathfrak{p}}\cdot K_-} = e^{\mathfrak{q}\cdot K_+} e^{-2\log(\cosh(2r))K_0} e^{-\bar{\mathfrak{q}}\cdot K_-}$$

where $\mathfrak{p} = r\sigma_0 e^{i\theta\sigma(\hat{n})} = re^{i\theta\sigma(\hat{n})}$, $\mathfrak{q} = \tanh(r)e^{i\theta\sigma(\hat{n})}$.

Proof To show the statement we look for $\alpha, \mathfrak{b}, \gamma \in \mathbb{H}$ such that $\alpha, \mathfrak{b}, \gamma$ mutually commuting, namely they belong to the same slice, such that the formula

$$e^{\mathfrak{p}\cdot K_+ - \bar{\mathfrak{p}}\cdot K_-} = e^{\alpha\cdot K_+} e^{\mathfrak{b}\cdot K_0} e^{\gamma\cdot K_-} \tag{4.1}$$

holds. We set $A = \mathfrak{p}\cdot K_+ - \bar{\mathfrak{p}}\cdot K_-$ and using the Baker-Campbell-Hausdorff formula we compute $e^A K_0 e^{-A}$. We have that

$$\begin{aligned}
&[A, K_0] = -(\mathfrak{p}\cdot K_+ + \bar{\mathfrak{p}}\cdot K_-),\\
&[A, [A, K_0]] = 2^2|\mathfrak{p}|^2 K_0\\
&[A, [A, [A, K_0]]] = -4|\mathfrak{p}|^2(\mathfrak{p}\cdot K_+ + \bar{\mathfrak{p}}\cdot K_-)\\
&[A, [A, [A, [A, K_0]]]] = 2^4|\mathfrak{p}|^4 K_0\\
&[A, [A, [A, [A, [A, K_0]]]]] = -2^4|\mathfrak{p}|^4 K_0(\mathfrak{p}\cdot K_+ + \bar{\mathfrak{p}}\cdot K_-)\\
&\dots\dots\dots\dots
\end{aligned}$$

from which we deduce

$$e^A K_0 e^{-A} = -\sinh(2r)(e^{i\theta\sigma(\hat{n})}\cdot K_+ + e^{-i\theta\sigma(\hat{n})}\cdot K_-) + \cosh(2r)K_0. \tag{4.2}$$

We note that the computations mimic the analogous computations in the classical case, since K_+, K_-, K_0 belong to $su(1,1)$, the quaternionic variables behaves like a variable commuting with the operators with respect to the left multiplication and the various quaternionic variables are assumed to be mutually commuting. Thus, reasoning as in the classical case, one obtains

$$e^A K_- e^{-A} = \sinh^2(r)e^{i2\theta\sigma(\hat{n})}\cdot K_+ + \cosh^2(r)K_- - \sinh(2r)e^{i\theta\sigma(\hat{n})}\cdot K_0. \tag{4.3}$$

Let us denote by B the operator on the right hand side of formula (4.1) and let us compute BK_0B^{-1} and BK_-B^{-1}. To compute the first one we start first by computing $e^{\alpha\cdot K_+}K_0e^{-\alpha\cdot K_+}$. A standard computation shows that

$$e^{\gamma\cdot K_-}K_0 e^{-\gamma\cdot K_-} = \gamma\cdot K_-.$$

Then one computes $e^{\mathfrak{b}\cdot K_0}(\gamma\cdot K_-)e^{-\mathfrak{b}\cdot K_0}$ and finally $e^{\alpha\cdot K_+}$. The result is

$$BK_0B^{-1} = (1 - 2\alpha\gamma e^{-\mathfrak{b}})\cdot K_0 + \gamma e^{-\mathfrak{b}}\cdot K_- - \alpha(1 - e^{-\mathfrak{b}}\alpha\gamma)\cdot K_+. \tag{4.4}$$

Reasoning in a similar way, and basically using the same computations as in the classical case, we obtain

$$BK_-B^{-1} = -2\alpha e^{-\mathfrak{b}}\cdot K_0 + e^{-\mathfrak{b}}\cdot K_- + e^{-\mathfrak{b}}\alpha^2\cdot K_+ \tag{4.5}$$

By comparing the coefficients obtained in (4.2), (4.3) and in (4.4), (4.5) one obtains $\alpha = e^{i\theta\sigma(\hat{n})}\tanh(r)$, $\mathfrak{b} = -2\log(\cosh(r))$, $\gamma = -e^{-i\theta\sigma(\hat{n})}\tanh(r)$ and the statement follows. □

We will need the above disentanglement formula to realize the connection between squeezed states and Hermite polynomials in a quaternion slice. The quaternionic Hermite polynomials are given by

$$H_n(\mathfrak{q}) = n!\sum_{m=0}^{[n/2]}\frac{(-1)^m(2\mathfrak{q})^{n-2m}}{m!(n-2m)!}, \quad \text{for all } \mathfrak{q}\in\mathbb{H}. \tag{4.6}$$

The following Proposition connects the squeezed states to Hermite polynomials in a quaternion slice that the squeezed basis vectors are essentially the Hermite polynomials times an exponential function.

Proposition 4.4 *For any* $\mathfrak{p} \in \mathbb{C}_I$ *with* $\mathfrak{p} \neq 0$,

$$\mathfrak{r} = \frac{\tanh(|\mathfrak{p}|)}{|\mathfrak{p}|}\mathfrak{p}, \tag{4.7}$$

then the squeezed basis vector, in the Bargmann analytic (in $\mathbb{C}_I$*) representation are given in terms of the complex Hermite polynomials by the expression,*

$$\Phi_n^{\mathfrak{p}}(\mathfrak{q}) = (S(\mathfrak{p})\Phi_n)(\mathfrak{q}) = \frac{1}{\sqrt{n!}}(1-|\mathfrak{r}|^2)^{\frac{1}{4}}\left[\frac{\overline{\mathfrak{r}}}{2}\right]^{\frac{n}{2}} e^{\frac{\mathfrak{r}}{2}\mathfrak{q}^2} H_n\left(\left[\frac{1}{2}(1-|\mathfrak{r}|^2)\overline{\mathfrak{r}}^{-1}\right]^{\frac{1}{2}}\mathfrak{q}\right). \tag{4.8}$$

Proof We have, from (4.7), $\log(1-|\mathfrak{r}|^2) = -2\log\cosh|\mathfrak{r}|$. So the above Proposition (4.3) enables us to write the squeeze operator $S(\mathfrak{p})$ as

$$S(\mathfrak{p}) = e^{\frac{\mathfrak{r}}{2}\mathfrak{q}^2}\, e^{\frac{1}{2}\log(1-|\mathfrak{r}|^2)(\mathfrak{q}\partial_s + \frac{1}{2}I_{\mathfrak{H}_r^B})}\, e^{-\frac{\overline{\mathfrak{r}}}{2}\partial_s^2}. \tag{4.9}$$

Now the basis vector $\Phi_n(\mathfrak{q}) = \dfrac{\mathfrak{q}^n}{\sqrt{n!}}$. Further left slice regular derivative of a regular function is regular, and for $\{\mathfrak{a}_m\} \subseteq \mathbb{H}$, we have, for a right regular power series (see, for example [19]),

$$\partial_s\left(\sum_{m=0}^{\infty}\mathfrak{a}_m\mathfrak{q}^m\right) = \sum_{m=0}^{\infty} m\mathfrak{a}_m\mathfrak{q}^{m-1}. \tag{4.10}$$

Thus, by doing a right regular power series expansion, we easily obtain,

$$e^{-\frac{\overline{\mathfrak{r}}}{2}\partial_s^2}\Phi_n(\mathfrak{q}) = e^{-\frac{\overline{\mathfrak{r}}}{2}\partial_s^2}\left(\frac{\mathfrak{q}^n}{\sqrt{n!}}\right) = n!\sum_{m=0}^{[n/2]}\frac{(-1)^m\left(\frac{\overline{\mathfrak{r}}}{2}\right)^m(\mathfrak{q})^{n-2m}}{m!(n-2m)!}. \tag{4.11}$$

One can note that, for any integer k, since $\mathfrak{q}\partial_s\mathfrak{q}^k = k\mathfrak{q}^k$,

$$e^{\frac{1}{2}\log(1-|\mathfrak{r}|^2)(\mathfrak{q}\partial_s + \frac{1}{2}I_{\mathfrak{H}_r^B})}\mathfrak{q}^k = (\sqrt{1-|\mathfrak{r}|^2})^{k+\frac{1}{2}}\mathfrak{q}^k. \tag{4.12}$$

Combining (4.9)–(4.12), and noting (4.6), the result (4.8) follows. □

5 Conclusion

Using the left multiplication on a right quaternionic Hilbert space we have defined unitary squeeze operator. Pure squeezed states have been obtained, with all the desired properties, analogous to their complex counterpart. Even though we have defined squeezed states with the aid of displacement operator and the squeeze operator, on the whole set of quaternions, the non-commutativity of quaternions prevented us in getting desired expectation values and variances. Even though it is a technical issue, there is no known technique developed yet to overcome this difficulty. In this regard, the only way out of this difficulty is to consider quaternionic slice-wise approach. We have defined squeezed states on quaternion slices and computed the expectation values of the quadrature operators. We have also proved a quaternionic disentanglement formula.

In the application point of view squeezed states have several applications, particularly in coding and transmission of information through optical devices. These aspects are well explained for example in [3, 10, 22] and the many references therein. Since we have used the matrix representation of quaternions, the squeezed states obtained in this note appear as matrix states. Further these states involve all four variables of quaternions. These features may give advantages in applications.

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References

1. Adler, S.L.: Quaternionic Quantum Mechanics and Quantum Fields. Oxford University Press, New York (1995)
2. Adler, S.L., Millard, A.C.: Coherent states in quaternionic quantum mechanics. J. Math. Phys. **38**, 2117–26 (1997)
3. Ali, S.T., Antoine, J.-P., Gazeau, J.-P.: Coherent States, Wavelets and Their Generalizations, 2nd edn. Springer, New York (2014)
4. Alpay, D., Colombo, F., Kimsey, D.P.: The spectral theorem for quaternionic unbounded normal operators based on the S-spectrum. J. Math. Phys. **57**, 023503 (2016)
5. Alpay, D., Colombo, F., Sabadini, I., Salomon, G.: The Fock Space in the Slice Hyperholomorphic Setting, Hypercomplex Analysis: New Perspective and Applications, Trends in Mathematics, pp. 43–59. Birkhüser, Basel (2014)
6. Alpay, D., Colombo, F., Sabadini, I.: Slice Hyperholomorphic Schur Analysis. Besel, Birkhauser (2016)
7. Birkhoff, G., von Neumann, J.: The logic of quantum mechanics. Ann. Math. **37**, 823–843 (1936)
8. Brian, C.: Hall, Quantum Theory for Mathematicians. Springer, New York (2013)
9. Colombo, F., Sabadini, I., Struppa, D.C.: Noncommutative Functional Calculus. Birkhäuser, Basel (2011)
10. Gazeau, J.-P.: Coherent States in Quantum Physics. Wiley, Berlin (2009)
11. Gentili, G., Struppa, D.C.: A new theory of regular functions of a quaternionic variable. Adv. Math. **216**, 279–301 (2007)
12. Ghiloni, R., Moretti, W., Perotti, A.: Continuous slice functional calculus in quaternionic Hilbert spaces. Rev. Math. Phys. **25**, 1350006 (2013)
13. Gülebeck, K., Habetha, K., Spröbig, W.: Holomorphic Functions in the Plane and n-Dimensional Spaces. Birkhäuser, Basel (2008)
14. Loudon, R., Knight, P.L.: Squeezed light. J. Mod. Opt. **34**, 709–759 (1987)
15. Loudon, R.: The Quantum Theory of Light, 3rd edn. Oxford University Press, New York (2000)
16. Muraleetharan, B., Thirulogasanthar, K.: Coherent states on quaternion slices and a measurable field of Hilbert spaces. J. Geom. Phys. **110**, 233–247 (2016)
17. Muraleetharan, B., Thirulogasanthar, K.: Coherent state quantization of quaternions. J. Math. Phys. **56**, 083510 (2015)

18. Muraleetharam, B., Thirulogasanthar, K., Sabadini, I.: A representation of Weyl-Heisenberg algebra in the quaternionic setting. Ann. Phys. **365**, 180–213 (2017)
19. Thirulogasanthar, K., Twareque Ali, S.: Regular subspaces of a quaternionic Hilbert space from quaternionic Hermite polynomials and associated coherent states. J. Math. Phys. **54**, 013506 (2013)
20. Thirulogasanthar, K., Honnouvo, G., Krzyzak, A.: Coherent states and Hermite polynomials on Quaternionic Hilbert spaces. J. Phys.A: Math. Theor. **43**, 385205 (2010)
21. Viswanath, K.: Normal operators on quaternionic Hilbert spaces. Trans. Am. Math. Soc. **162**, 337–350 (1971)
22. Youen, Y.P.: Two photon coherent states of the radiation field. Phys. Rev. A. **13**, 2226–43 (1976)
23. Zhang, F.: Quaternions and matrices of quaternions. Linear Algebra Appl. **251**, 21–57 (1997)