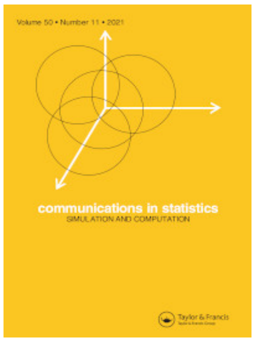

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Modified almost unbiased Liu estimator in logistic regression

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ABSTRACT
This paper focuses on introducing a new parameter estimator to the logistic regression model when the multicollinearity presents. The proposed estimator called Modified almost unbiased logistic Liu estimator (MAULLE) is obtained by composing the Liu estimator and the almost unbiased Liu estimator. Further, conditions for the superiority of the new estimator over some existing estimators were derived with respect to mean square error (MSE) and scalar mean square error (SMSE) sense. A Monte Carlo simulation study was carried out to compare the performance of the proposed estimator with some existing estimators in the scalar mean square error sense, and a real data example was discussed to illustrate the theoretical results.

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Logistic Regression; Multicollinearity; Liu estimator; Modified almost unbiased logistic Liu estimator; Mean square error

1. Introduction

Consider the logistic regression model

$$y_i = \pi_i + \varepsilon_i, \quad i = 1, ..., n \tag{1}$$

which follows Binary distribution with parameter π_i as

$$\pi_i = \frac{\exp(x_i{}'\beta)}{1 + \exp(x_i{}'\beta)}, \tag{2}$$

where x_i is the i^{th} row of X, which is an $n \times p$ data matrix with p explanatory variables and β is a $p \times 1$ vector of coefficients, ε_i are independent with mean zero and variance $\pi_i(1-\pi_i)$ of the response y_i. To estimate the regression parameter β, the maximum likelihood method is commonly used. The likelihood function of the logistic regression model (1)

$$L(\beta) = \prod_{i=1}^{n} \pi_i^{y_i}(1-\pi_i)^{(1-y_i)} \tag{3}$$

Consequently, the log-likelihood function of the logistic regression model can be written as

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$$\ell(\beta) = \sum_{i=1}^{n} y_i ln(\pi_i) + \sum_{i=1}^{n} (1-y_i) ln(1-\pi_i). \tag{4}$$

The likelihood estimating equation becomes

$$\frac{\partial \ell(\beta)}{\partial \beta} = \sum_{i=1}^{n} [y_i - \pi_i] x_i = 0. \tag{5}$$

Then the maximum likelihood estimator (MLE) of β can be obtained by solving the above estimating Eq. (5). Since the Eq. (5) is nonlinear in parameter β, one can use the Newton-Raphson iterative algorithm to estimate the parameter β.

$$\hat{\beta}_{MLE} = C^{-1} X' \hat{W} Z, \tag{6}$$

where $C = X'\hat{W}X$; Z is the column vector with i^{th} element equals $logit(\hat{\pi}_i) + \frac{y_i - \hat{\pi}_i}{\hat{\pi}_i(1-\hat{\pi}_i)}$ and $\hat{W} = diag[\hat{\pi}_i(1-\hat{\pi}_i)]$. Note that $\hat{\beta}_{MLE}$ is asymptotically unbiased for β and its covariance matrix is

$$Cov\left(\hat{\beta}_{MLE}\right) = \{X'\hat{W}X\}^{-1}. \tag{7}$$

The MSE and SMSE of $\hat{\beta}_{MLE}$ are

$$\begin{aligned} MSE\left[\hat{\beta}_{MLE}\right] &= Cov\left[\hat{\beta}_{MLE}\right] + B\left[\hat{\beta}_{MLE}\right] B'\left[\hat{\beta}_{MLE}\right] \\ &= \{X'\hat{W}X\}^{-1} \\ &= C^{-1} \end{aligned} \tag{8}$$

and

$$SMSE\left[\hat{\beta}_{MLE}\right] = tr[C^{-1}] \tag{9}$$

Since C is a positive definite matrix, there exists an orthogonal matrix P such that $P'CP = \Delta = diag(\lambda_1, \lambda_2, ..., \lambda_p)$, where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p > 0$ are the ordered eigen values of C. Then

$$SMSE\left[\hat{\beta}_{MLE}\right] = \sum_{j=1}^{p} \frac{1}{\lambda_j} \tag{10}$$

The maximum likelihood estimator performs better if the explanatory variables are uncorrelated. However, there are some situations where the explanatory variables are correlated, which is called as multicollinearity, then the maximum likelihood method produces inefficient estimates.

In order to reduce the effect of multicollinearity in logistic regression, many alternative estimators have been proposed in the literature. These estimators are mainly categorized into two cases, such as estimators based only on the sample information and the estimators based on the sample and prior information. The estimators based only on the sample information are Logistic Ridge Estimator (LRE) by Schaefer et al. (1984), the Principal Component Logistic Estimator (PCLE) by Aguilera et al. (2006), the Modified Logistic Ridge Estimator (MLRE) by Nja et al. (2013), the Logistic Liu Estimator (LLE) by Mansson et al. (2012), the Liu-Type Logistic Estimator (LTLE) by Inan and Erdogan

(2013), the Almost Unbiased Ridge Logistic Estimator (AURLE) by Wu and Asar (2016), the Almost Unbiased Logistic Liu Estimator (AULLE) by Xinfeng (2015), and the Optimal Generalized Logistic Estimator (OGLE) by Varathan and Wijekoon (2018a).

The estimators based on the sample and priori available information are the Restricted Maximum Likelihood Estimator (RMLE) by Duffy and Santner (1989), the Restricted Logistic Liu Estimator (RLLE) by Şiray et al. (2015), the Modified Restricted Liu Estimator by Wu (2016), the Restricted Logistic Ridge Estimator (RLRE) by Asar et al. (2017a), the Restricted Liu-Type Logistic Estimator (RLTLE) by Asar et al. (2017b), the Restricted Almost Unbiased Ridge logistic Estimator (RAURLE) by Varathan and Wijekoon (2016a), the Stochastic Restricted Maximum Likelihood Estimator (SRMLE) by Nagarajah and Wijekoon (2015), the Stochastic Restricted Ridge Maximum Likelihood Estimator (SRRMLE) by Varathan and Wijekoon (2016b), the Stochastic Restricted Liu Maximum Likelihood Estimator (SRLMLE) by Varathan and Wijekoon (2016c), and the Stochastic restricted Liu-type logistic estimator (SRLTLE) by Varathan and Wijekoon (2018b).

Note that none of the above existing estimators are always superior. By adding different weights to the MLE, some of the existing estimators based on sample information were developed to reduce multicollinearity. For example, the weight $(I+kC^{-1})^{-1}$ is added to MLE to obtain LRE, the weight $(C+I)^{-1}(C+\mathrm{d}I)$ is added to MLE to obtain LLE, and the weight $[I-(1-d)^2(C+I)^{-2}]$ is added to MLE to obtain AULLE. This motivated us to propose a new estimator by combining AULLE and LLE by means of giving more weight to MLE, and the proposed new estimator is named as Modified almost unbiased logistic Liu estimator (MAULLE). Further, we compare the performance of the proposed estimator MAULLE with the existing estimators MLE, LLE and AULLE in the mean square error sense.

The organization of the paper is as follows: The construction of the proposed estimator and the properties are given in Sec. 2. In Sec. 3, the conditions for superiority of the proposed estimator MAULLE over the existing estimators MLE, LLE, and AULLE are derived with respect to mean square error (MSE) and scalar mean square error (SMSE) criterions. The detail Monte Carlo simulation study is given to investigate the performance of the proposed estimator in the scalar mean squared error (SMSE) sense in Sec. 4. A real data application is discussed in Sec. 5. Finally, some conclusive remarks are given in Sec. 6.

2. Construction of proposed estimator

Among the various estimators proposed in the literature to reduce the effect of multicollinearity in logistic regression model, here we consider the Logistic Liu estimator (LLE) (Mansson et al. 2012) and the Almost Unbiased Logistic Liu Estimator (AULLE) (Xinfeng 2015) to construct the proposed estimator.

2.1. Logistic Liu estimator (LLE)

The Logistic Liu estimator (LLE) (Mansson et al. 2012) is defined as

$$\hat{\beta}_{LLE} = Z_d\hat{\beta}_{MLE} \tag{11}$$

where $Z_d = (C+I)^{-1}(C+dI)$, $0 < d < 1$

The expectation, covariance, bias, mean square error and scalar mean square error of LLE are given by

$$E\left[\hat{\beta}_{LLE}\right] = Z_d\beta \tag{12}$$

$$D\left[\hat{\beta}_{LLE}\right] = Z_d C^{-1} Z_d{}' \tag{13}$$

$$B\left[\hat{\beta}_{LLE}\right] = [Z_d - I]\beta \tag{14}$$

$$MSE\left[\hat{\beta}_{LLE}\right] = Z_d C^{-1} Z_d{}' + [Z_d - I]\beta\beta'[Z_d - I]' \tag{15}$$

and

$$SMSE\left[\hat{\beta}_{LLE}\right] = tr\{Z_d C^{-1} Z_d'\} + \beta'[Z_d - I]'[Z_d - I]\beta = \sum_{j=1}^{p} \frac{(\lambda_j + d)^2}{\lambda_j(\lambda_j + 1)^2} + \sum_{j=1}^{p} \frac{(d-1)^2\alpha_j^2}{(\lambda_j + 1)^2} \tag{16}$$

where α_j is the jth element of $P'\beta$, $j = 1, \ldots p$.

2.2. Almost Unbiased Logistic Liu estimator (AULLE)

The Almost unbiased logistic Liu estimator (AULLE) (Xinfeng 2015) is defined as

$$\hat{\beta}_{AULLE} = W_d\hat{\beta}_{MLE}; \tag{17}$$

where $W_d = [I - (1-d)^2(C+I)^{-2}], 0 < d < 1$.

The expectation, covariance, bias, mean square error and scalar mean square error of AULLE are given by

$$E\left[\hat{\beta}_{AULLE}\right] = W_d\beta \tag{18}$$

$$D\left[\hat{\beta}_{AULLE}\right] = W_d C^{-1} W_d{}' \tag{19}$$

$$B\left[\hat{\beta}_{AULLE}\right] = [W_d - I]\beta \tag{20}$$

$$MSE\left[\hat{\beta}_{AULLE}\right] = W_d C^{-1} W_d{}' + [w_d - I]\beta\beta'[W_d - I]' \tag{21}$$

and

$$\begin{aligned} SMSE\left[\hat{\beta}_{AULLE}\right] &= tr\{W_d C^{-1} W_d'\} + \beta'[W_d - I]'[W_d - I]\beta \\ &= \sum_{j=1}^{p} \frac{(\lambda_j + d)^2(\lambda_j + 2 - d)^2}{\lambda_j(\lambda_j + 1)^4} + \sum_{j=1}^{p} \frac{(d-1)^4\alpha_j^2}{(\lambda_j + 1)^4} \end{aligned} \tag{22}$$

2.3. Proposed estimator

By adopting $\hat{\beta}_{LLE}$ in place of $\hat{\beta}_{MLE}$ in (17), we propose a new estimator which is called as the Modified almost unbiased logistic Liu estimator (MAULLE) and defined as

$$\hat{\beta}_{MAULLE} = W_d\hat{\beta}_{LLE} = W_d Z_d\hat{\beta}_{MLE} = F_d\hat{\beta}_{MLE} \tag{23}$$

where

$$F_d = W_d Z_d \; = \left[I - (1-d)^2 (C+I)^{-2}\right]\left[(C+I)^{-1}(C+dI)\right] \tag{24}$$

The asymptotic properties of $\hat{\beta}_{MAULLE}$ are

$$E\left[\hat{\beta}_{MAULLE}\right] = E\left[F_d \hat{\beta}_{MLE}\right] = F_d \beta, \tag{25}$$

$$D\left(\hat{\beta}_{MAULLE}\right) = Cov\left(\hat{\beta}_{MAULLE}\right) \; = Cov\left(F_d \hat{\beta}_{MLE}\right) \; = F_d C^{-1} F_d{}', \tag{26}$$

and

$$Bias\left(\hat{\beta}_{MAULLE}\right) = E\left[\hat{\beta}_{MAULLE}\right] - \beta \; = [F_d - I]\beta \tag{27}$$

Consequently, the mean square error matrix and scalar mean squared error can be obtained as,

$$\begin{aligned} MSE\left(\hat{\beta}_{MAULLE}\right) &= D\left(\hat{\beta}_{MAULLE}\right) + Bias\left(\hat{\beta}_{MAULLE}\right) Bias\left(\hat{\beta}_{MAULLE}\right)' \\ &= F_d C^{-1} F_d{}' + (F_d - I)\beta\beta'(F_d - I)' \end{aligned} \tag{28}$$

and

$$SMSE\left(\hat{\beta}_{MAULLE}\right) = tr\left[F_d C^{-1} F_d{}'\right] + \beta'(F_d - I)'(F_d - I)\beta \tag{29}$$

where

$$\begin{aligned} F_d &= W_d Z_d \\ &= [I - (1-d)^2 (C+I)^{-2}]\left[(C+I)^{-1}(C+dI)\right] \\ &= [(C+dI)(C+2I-dI)(C+I)^{-3}(C+dI)] \end{aligned} \tag{30}$$

Then

$$\begin{aligned} tr[F_d C^{-1} F_d{}'] &= tr[F_d{}' F_d C^{-1}] \\ &= tr[(C+dI)(C+I)^{-3}(C+2I-dI)(C+dI)^2 \\ &\quad (C+2I-dI)(C+I)^{-3}(C+dI)]C^{-1} \\ &= tr[P'(\Delta+dI)(\Delta+I)^{-3}(\Delta+2I-dI)(\Delta+I)^{-3}][(\Delta+dI)^2 \\ &\quad (\Delta+2I-dI)(\Delta+I)^{-3}(\Delta+dI)\Delta^{-1}P] \\ &= \sum_{j=1}^{p} \frac{(\lambda_j+d)^4(\lambda_j+2-d)^2}{\lambda_j(\lambda_j+1)^6} \end{aligned} \tag{31}$$

Now consider

$$F_d - I = (C+dI)(C+2I-dI)(C+I)^{-3}(C+dI) - I$$

Consequently, $\beta'(F_d - I)'(F_d - I)\beta$ can be expressed as

$$\begin{aligned} \beta'(F_d - I)'(F_d - I)\beta &= \sum_{j=1}^{p} \left\{ \frac{(\lambda_j+d)^2(\lambda_j+2-d)}{(\lambda_j+1)^3} - 1 \right\}^2 \alpha_j^2 \\ &= \sum_{j=1}^{p} \left\{ \frac{(\lambda_j^2 + 2d\lambda_j + d^2)(\lambda_j+2-d) - (\lambda_j+1)^3}{(\lambda_j+1)^3} \right\}^2 \alpha_j^2 \\ &= \sum_{j=1}^{p} \frac{\left\{(d-1)\lambda_j^2 - (d-1)(d-3)\lambda_j + (2d^2 - d^3 - 1)\right\}^2}{(\lambda_j+1)^6} \alpha_j^2 \end{aligned} \tag{32}$$

Therefore,

$$SMSE(\hat{\beta}_{MAULLE}) = \sum_{j=1}^{p} \frac{(\lambda_j + d)^4 (\lambda_j + 2-d)^2}{\lambda_j (\lambda_j + 1)^6} + \sum_{j=1}^{p} \frac{\left\{(d-1)\lambda_j^2 - (d-1)(d-3)\lambda_j + (2d^2 - d^3 - 1)\right\}^2}{(\lambda_j + 1)^6} \alpha_j^2 \tag{33}$$

3. Comparison of estimators

One of the most popular method used to evaluate the performance of the two competing estimators is the Mean square error criterion. The mean square error matrix for an estimator $\hat{\beta}$ of the parameter β is defined by

$$\begin{aligned} MSE(\hat{\beta}, \beta) &= E\left[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'\right] \\ &= D(\hat{\beta}) + B(\hat{\beta}) B'(\hat{\beta}) \end{aligned} \tag{34}$$

where $D(\hat{\beta})$ is the dispersion matrix, and $B(\hat{\beta}) = E(\hat{\beta}) - \beta$ denotes the bias vector.

For two given estimators $\hat{\beta}_1$ and $\hat{\beta}_2$, the estimator $\hat{\beta}_2$ is said to be superior to $\hat{\beta}_1$ under the MSE criterion if and only if

$$M\left(\hat{\beta}_1, \hat{\beta}_2\right) = MSE\left(\hat{\beta}_1, \beta\right) - MSE\left(\hat{\beta}_2, \beta\right) \geq 0 \tag{35}$$

However, in practical situations the scalar mean square error (SMSE) is preferred to compare the estimators. The scalar mean square Error of the estimator $\hat{\beta}$ can be defined as

$$SMSE(\hat{\beta}, \beta) = trace\left[MSE(\hat{\beta}, \beta)\right] \tag{36}$$

For two given estimators $\hat{\beta}_1$ and $\hat{\beta}_2$, the estimator $\hat{\beta}_2$ is said to be superior to $\hat{\beta}_1$ under the SMSE criterion if and only if

$$SMSE\left(\hat{\beta}_1, \beta\right) - SMSE\left(\hat{\beta}_2, \beta\right) \geq 0. \tag{37}$$

To examine the superiority of the proposed estimator MAULLE with the existing estimators MLE, LLE and AULLE, in this section we use the following two lemmas.

Lemma 1: (Farebrother 1976). Let M be a positive definite matrix, namely $M > 0$, and let α be some vector, then $M - \alpha\alpha' \geq 0$ if and only if $\alpha' M^{-1} \alpha \leq 1$.

Lemma 2: (Trenkler and Toutenburg 1990). Let $\tilde{\beta}_j = A_j y$, $j = 1, 2$ be two competing homogeneous linear estimators of β. Suppose that $D = Cov(\tilde{\beta}_1) - Cov(\tilde{\beta}_2) > 0$, where $Cov(\tilde{\beta}_j)$, $j = 1, 2$ denotes the covariance matrix of $\tilde{\beta}_j$. Then $\Delta(\tilde{\beta}_1, \tilde{\beta}_2) = MSEM(\tilde{\beta}_1) - MSEM(\tilde{\beta}_2) \geq 0$ if and only if $d_2'(D + d_1' d_1)^{-1} d_2 \leq 1$, where $MSEM(\tilde{\beta}_j)$, d_j; $j = 1, 2$ denote the Mean Square Error Matrix and bias vector of $\tilde{\beta}_j$, respectively.

3.1. MSE comparison of estimators

In this subsection, we compare the mean square error of the proposed estimator with the existing estimators MLE, LLE, and AULLE.

3.1.1. Superiority of the MAULLE over MLE

In order to compare the performance of MAULLE with MLE, we consider the MSE difference

$$\begin{aligned} MSE\left(\hat{\beta}_{MLE}\right) - MSE\left(\hat{\beta}_{MAULLE}\right) &= C^{-1} - \left\{F_d C^{-1} F_d{}' + (F_d - I)\beta\beta'(F_d - I)'\right\} \\ &= \left\{C^{-1} - F_d C^{-1} F_d{}'\right\} - (F_d - I)\beta\beta'(F_d - I)' \\ &= M_1 - \alpha_1 \alpha_1{}' \end{aligned} \tag{38}$$

where $M_1 = C^{-1} - F_d C^{-1} F_d{}'$ and $\alpha_1 = (F_d - I)\beta$.

To check M_1 is positive definite matrix, we consider

$$\begin{aligned} M_1 &= C^{-1} - F_d C^{-1} F_d{}' \\ &= C^{-1} - [I - (1-d)^2 (C+I)^{-2}][(C+I)^{-1}(C+dI)]C^{-1}[(C+dI)(C+I)^{-1}] \\ &\quad [I - (1-d)^2 (C+I)^{-2}] \\ &= \{C^{-1} - (C+I)^{-1}(C+dI)C^{-1}(C+dI)(C+I)^{-1}\} \\ &\quad + \{(1-d)^2 (C+I)^{-1}(C+dI)C^{-1}(C+dI)(C+I)^{-3}\} \\ &\quad + \{(1-d)^2 (C+I)^{-3}(C+dI)C^{-1}(C+dI)(C+I)^{-1}\} \\ &\quad - \{(1-d)^4 (C+I)^{-3}(C+dI)C^{-1}(C+dI)(C+I)^{-3}\} \end{aligned} \tag{39}$$

Now consider the term

$$\begin{aligned} & C^{-1} - (C+I)^{-1}(C+dI)C^{-1}(C+dI)(C+I)^{-1} \\ &= (C+I)^{-1}[(C+I)C^{-1}(C+I) - (C+dI)C^{-1}(C+dI)](C+I)^{-1} \\ &= (C+I)^{-1}[C^{-1}(1-d^2) + 2(1-d)I](C+I)^{-1} \\ & > 0 \end{aligned}$$

Next consider

$$\begin{aligned} & \{(1-d)^2 (C+I)^{-3}(C+dI)C^{-1}(C+dI)(C+I)^{-1}\} \\ & \quad - \{(1-d)^4 (C+I)^{-3}(C+dI)C^{-1}(C+dI)(C+I)^{-3}\} \\ &= (1-d)^2 (C+I)^{-3}(C+dI)C^{-1}(C+dI)(C+I)^{-3}[(C+I)^{-2} - (1-d)^2 I] \\ &= (1-d)^2 (C+I)^{-3}(C+dI)C^{-1}(C+dI)(C+I)^{-3}[C(C+2I) + d(2-d)I] \\ & > 0 \end{aligned}$$

Further it is obvious that $(1-d)^2(C+I)^{-1}(C+dI)C^{-1}(C+dI)(C+I)^{-3}$ is a positive definite matrix. So, M_1 is a positive definite matrix. By applying Lemma 1, we can get, $MSE(\hat{\beta}_{MLE}) - MSE(\hat{\beta}_{MAULLE}) \geq 0$ *iff* $\beta'(F_d - I)'[C^{-1} - F_d C^{-1} F_d{}']^{-1}(F_d - I)\beta \leq 1$. Therefore it can be concluded that MAULLE is superior to MLE in the MSE sense if and only if $\beta'(F_d - I)'[C^{-1} - F_d C^{-1} F_d{}']^{-1}(F_d - I)\beta \leq 1$.

3.1.2. Superiority of the MAULLE over LLE

Consider the MSE difference of the estimators MAULLE and LLE

$$\begin{aligned} MSE\left(\hat{\beta}_{LLE}\right)-MSE\left(\hat{\beta}_{MAULLE}\right) &= \left\{Z_dC^{-1}Z_d' + (Z_d-I)\beta\beta'(Z_d-I)'\right\} \\ &\quad -\left\{F_dC^{-1}F_d' + (F_d-I)\beta\beta'(F_d-I)'\right\} \\ &= \{Z_dC^{-1}Z_d'-F_dC^{-1}F_d'\} \\ &\quad +\left\{(Z_d-I)\beta\beta'(Z_d-I)'-(F_d-I)\beta\beta'(F_d-I)'\right\} \end{aligned} \tag{40}$$

Now consider,

$$\begin{aligned} D\left(\hat{\beta}_{LLE}\right)-D\left(\hat{\beta}_{MAULLE}\right) &= Z_dC^{-1}Z_d'-F_dC^{-1}F_d' \\ &= Z_dC^{-1}Z_d'-W_dZ_dC^{-1}Z_d'W_d' \\ &= Z_dC^{-1}Z_d'-\left[I-(1-d)^2(C+I)^{-2}\right][Z_dC^{-1}Z_d''] \\ &\quad [I-(1-d)^2(C+I)^{-2}]' \\ &= (1-d)^4(C+I)^{-2}Z_dC^{-1}Z_d'(C+I)^{-2} \\ &= D_1(say) \\ &> 0 \end{aligned} \tag{41}$$

By applying Lemma 2, we can get

$$MSE\left(\hat{\beta}_{LLE}\right)-MSE\left(\hat{\beta}_{MAULLE}\right)\geq 0 \text{ iff} \beta'(F_d-I)'\left[D_1+(Z_d-I)\beta\beta'(Z_d-I)'\right]^{-1}(F_d-I)\beta\leq 1$$

Therefore one can say that the estimator MAULLE is superior to LLE in the MSE sense if and only if $\beta'(F_d-I)'[D_1+(Z_d-I)\beta\beta'(Z_d-I)']^{-1}(F_d-I)\beta \leq 1$.

3.1.3. Superiority of the MAULLE over AULLE

The MSE difference of the estimators MAULLE and AULLE

$$\begin{aligned} MSE\left(\hat{\beta}_{AULLE}\right)-MSE\left(\hat{\beta}_{MAULLE}\right) &= \{W_dC^{-1}W_d' + (W_d-I)\beta\beta'(W_d-I)'\} \\ &\quad -\{F_dC^{-1}F_d' + (F_d-I)\beta\beta'(F_d-I)'\} \\ &= \left\{W_dC^{-1}W_d'-F_dC^{-1}F_d'\right\} \\ &\quad +\{(W_d-I)\beta\beta'(W_d-I)'-(F_d-I)\beta\beta'(F_d-I)'\} \end{aligned} \tag{42}$$

Now consider,

$$D\left(\hat{\beta}_{AULLE}\right)-D\left(\hat{\beta}_{MAULLE}\right) = W_dC^{-1}W_d'-F_dC^{-1}F_d' = W_dC^{-1}W_d'-W_dZ_dC^{-1}Z_d'W_d' = W_d[C^{-1}-Z_dC^{-1}Z_d']W_d' \quad (43)$$

Next consider,

$$\begin{aligned} &C^{-1}-Z_dC^{-1}Z_d' \\ &= C^{-1}-(C+I)^{-1}(C+dI)C^{-1}(C+dI)'(C+I)^{-1} \\ &= (C+I)^{-1}[(C+I)C^{-1}(C+I)-(C+dI)C^{-1}(C+dI)](C+I)^{-1} \\ &= (C+I)^{-1}[2(1-d)I+(1-d^2)C^{-1}](C+I)^{-1} \\ &>0 \end{aligned}$$

This implies,

$$D_2 = D\left(\hat{\beta}_{AULLE}\right) - D\left(\hat{\beta}_{MAULLE}\right) = W_d\left[C^{-1} - Z_dC^{-1}Z_d'\right]W_d' > 0.$$

Now by applying Lemma 2, we can get

$$MSE\left(\hat{\beta}_{AULLE}\right)-MSE\left(\hat{\beta}_{MAULLE}\right)\geq 0 \text{ iff } \beta'(F_d-I)'\left[D_2+(W_d-I)\beta\beta'(W_d-I)'\right]^{-1}(F_d-I)\beta\leq 1$$

Therefore, it can be concluded that the estimator MAULLE is superior to AULLE in the MSE sense if and only if $\beta'(F_d - I)'[D_2 + (W_d - I)\beta\beta'(W_d - I)']^{-1}(F_d - I)\beta \leq 1$.

3.2. SMSE comparison of estimators

In this subsection, we compare the scalar mean square error of the proposed estimator with the existing estimators MLE, LLE, and AULLE.

3.2.1. MAULLE versus MLE

$$\begin{aligned} SMSE(\hat{\beta}_{MLE})-SMSE(\hat{\beta}_{MAULLE}) &= \sum_{j=1}^{p}\frac{1}{\lambda_j}-\left[\sum_{j=1}^{p}\frac{(\lambda_j+d)^4(\lambda_j+2-d)^2}{\lambda_j(\lambda_j+1)^6}\right. \\ &\quad \left.+\sum_{j=1}^{p}\frac{\left\{(d-1)\lambda_j^2-(d-1)(d-3)\lambda_j+(2d^2-d^3-1)\right\}^2}{(\lambda_j+1)^6}\alpha_j^2\right] \\ &= \sum_{j=1}^{p}\frac{(\lambda_j+1)^6-(\lambda_j+d)^4(\lambda_j+2-d)^2}{\lambda_j(\lambda_j+1)^6} \\ &\quad -\frac{\left\{(d-1)\lambda_j^3-(d-1)(d-3)\lambda_j^2+(2d^2-d^3-1)\lambda_j\right\}^2\alpha_j^2}{\lambda_j(\lambda_j+1)^6} \\ &= \Delta_1 \end{aligned} \quad (44)$$

Based on the above comparison, it can be noted that MAULLE is superior than MLE in the SMSE sense if and only if $\Delta_1 > 0$.

3.2.2. MAULLE versus LLE

$$
\begin{aligned}
MSE(\hat{\beta}_{LLE})-SMSE(\hat{\beta}_{MAULLE}) &= \left\{\sum_{j=1}^{p}\frac{(\lambda_j+d)^2}{\lambda_j(\lambda_j+1)^2}+\sum_{j=1}^{p}\frac{(d-1)^2\alpha_j^2}{(\lambda_j+1)^2}\right\} \\
&\quad -\left[\sum_{j=1}^{p}\frac{(\lambda_j+d)^4(\lambda_j+2-d)^2}{\lambda_j(\lambda_j+1)^6}\right. \\
&\quad \left.+\sum_{j=1}^{p}\frac{\{(d-1)\lambda_j^2-(d-1)(d-3)\lambda_j+(2d^2-d^3-1)\}^2}{(\lambda_j+1)^6}\alpha_j^2\right] \\
&= \sum_{j=1}^{p}\frac{1}{\lambda_j(\lambda_j+1)^6}\{(\lambda_j+1)^4(\lambda_j+d)^2-(\lambda_j+d)^4(\lambda_j+2-d)^2\} \\
&\quad +\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j+1)^6}\{(d-1)^2(\lambda_j+1)^4-[(d-1)\lambda_j^2 \\
&\quad -(d-1)(d-3)\lambda_j+(2d^2-d^3-1)]^2\} \\
&= \sum_{j=1}^{p}\frac{(\lambda_j+d)^2}{\lambda_j(\lambda_j+1)^6}\{(\lambda_j+1)^4-(\lambda_j+d)^2(\lambda_j+2-d)^2\} \\
&\quad +\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j+1)^6}\{(d-1)^2(\lambda_j+1)^4-[(d-1)\lambda_j^2 \\
&\quad -(d-1)(d-3)\lambda_j+(2d^2-d^3-1)]^2\} \\
&= \Delta_2
\end{aligned} \tag{45}
$$

From the above comparison, it can be concluded that MAULLE is superior than LLE in the SMSE sense if and only if $\Delta_2 > 0$.

3.2.3. MAULLE versus AULLE

$$
\begin{aligned}
SMSE(\hat{\beta}_{AULLE})-SMSE(\hat{\beta}_{MAULLE}) &= \left\{\sum_{j=1}^{p}\frac{(\lambda_j+d)^2(\lambda_j+2-d)^2}{\lambda_j(\lambda_j+1)^4}+\sum_{j=1}^{p}\frac{(d-1)^4\alpha_j^2}{(\lambda_j+1)^4}\right\} \\
&\quad -\left[\sum_{j=1}^{p}\frac{(\lambda_j+d)^4(\lambda_j+2-d)^2}{\lambda_j(\lambda_j+1)^6}\right. \\
&\quad \left.-\sum_{j=1}^{p}\frac{\{(d-1)\lambda_j^2-(d-1)(d-3)\lambda_j+(2d^2-d^3-1)\}^2}{(\lambda_j+1)^6}\alpha_j^2\right] \\
&= \sum_{j=1}^{p}\frac{(\lambda_j+d)^2(\lambda_j+2-d)^2[(\lambda_j+1)^2-(\lambda_j+d)^2]}{\lambda_j(\lambda_j+1)^6} \\
&\quad +\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j+1)^6}[(d-1)^4(\lambda_j+1)^2 \\
&\quad -[(d-1)\lambda_j^2-(d-1)(d-3)\lambda_j+(2d^2-d^3-1)]^2] \\
&= \sum_{j=1}^{p}\frac{(\lambda_j+d)^2(\lambda_j+2-d)^2(2\lambda_j+d+1)(1-d)}{\lambda_j(\lambda_j+1)^6} \\
&\quad +\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j+1)^6}[(d-1)^4(\lambda_j+1)^2 \\
&\quad -[(d-1)\lambda_j^2-(d-1)(d-3)\lambda_j+(2d^2-d^3-1)]^2] \\
&= \Delta_3
\end{aligned} \tag{46}
$$

Based on the above comparison, it can be said that MAULLE is superior than AULLE in the SMSE sense if and only if $\Delta_3 > 0$.

3.3. The choice of biasing parameter *d*

It is essential to find a suitable value for the biasing parameter d in the proposed estimator MAULLE. To obtain the optimal value of d, following Hoerl and Kennard (1970), we minimize the Scalar mean square error of the MAULLE with respect to d.

Consider the SMSE of MAULLE in (29),

$$SMSE\left(\hat{\beta}_{MAULLE}\right) = tr\left[F_d C^{-1} F_d'\right] + \beta'(F_d - I)'(F_d - I)\beta$$

By applying (31) and (32), one can obtain

$$SMSE(\hat{\beta}_{MAULLE}) = \sum_{j=1}^{p} \frac{(\lambda_j + d)^4(\lambda_j + 2 - d)^2}{\lambda_j(\lambda_j + 1)^6} + \sum_{j=1}^{p} \left\{ \frac{(\lambda_j + d)^2(\lambda_j + 2 - d)}{(\lambda_j + 1)^3} - 1 \right\}^2 \alpha_j^2$$

which can be simplified as

$$= \sum_{j=1}^{p} \frac{(\lambda_j + d)^4(\lambda_j + 2 - d)^2(1 + \lambda_j\alpha_j^2)}{\lambda_j(\lambda_j + 1)^6} + \sum_{j=1}^{p} \frac{\lambda_j\alpha_j^2[(\lambda_j + 1)^6 - 2(\lambda_j + d)^2(\lambda_j + 1)^3(\lambda_j + 2 - d)]}{\lambda_j(\lambda_j + 1)^6} \tag{47}$$

Now, differentiate (47) with respect to d and simplified as

$$\frac{\partial SMSE(\hat{\beta}_{MAULLE})}{\partial d} = \sum_{j=1}^{p} \frac{2(\lambda_j + d)(\lambda_j + 4 - 3d)[(1 + \lambda_j\alpha_j^2)(\lambda_j + d)^2(\lambda_j + 2 - d) - \lambda_j\alpha_j^2(\lambda_j + 1)^3]}{\lambda_j(\lambda_j + 1)^6} \tag{48}$$

Equate (48) to zero

$$\frac{\partial SMSE\left(\hat{\beta}_{MAULLE}\right)}{\partial d} = 0$$

implies

$$2(\lambda_j + d)(\lambda_j + 4 - 3d)[(1 + \lambda_j\alpha_j^2)(\lambda_j + d)^2(\lambda_j + 2 - d) - \lambda_j\alpha_j^2(\lambda_j + 1)^3] = 0, j = 1, 2, ..., p \tag{49}$$

Since the above Eq. (49) is in the form of polynomial of order five, one may obtain the estimated values of biasing parameter d, by using the help of computer software for known values of λ_j and α_j.

4. Monte Carlo simulation

In order to illustrate the behavior of the proposed estimator with the existing estimators: MLE, LLE, and AULLE, we perform a Monte Carlo simulation study by considering different levels of multicollinearity; $\rho = 0.7$, 0.8, 0.9 and 0.99. Further, in this study, we consider small, medium and large sample sizes; $n =$ 15, 50, and 100 respectively. The Scalar Mean Square Error (SMSE) is considered for the comparison of estimators. Following McDonald and Galarneau (1975) and Kibria (2003), we generate the explanatory variables as follows:

$$x_{ij} = (1-\rho^2)^{\frac{1}{2}} z_{ij} + \rho z_{i,p+1} \ , \ \ i = 1,2,...,n, j = 1,2,...,p \tag{50}$$

where z_{ij} are independent standard normal pseudo- random numbers and ρ is specified so that the theoretical correlation between any two explanatory variables is given by ρ^2. Four explanatory variables are generated using (50). The dependent variable y_i in (1) is obtained from the Bernoulli(π_i) distribution where $\pi_i = \frac{\exp(x_i{}'\beta)}{1+\exp(x_i{}'\beta)}$. The parameter values of $\beta_1, \beta_2, ..., \beta_p$ are chosen so that $\sum_{j=1}^{p} \beta_j^2 = 1$ and $\beta_1 = \beta_2 = ... = \beta_p$. Further, for the biasing parameter d we consider some selected values in the range $0 < d < 1$.

The simulation is repeated 1000 times by generating new pseudo- random numbers and the simulated SMSE values of the estimators are obtained using the following equation.

$$S\hat{M}SE\left(\hat{\beta}^*\right) = \frac{1}{1000}\sum_{r=1}^{1000}\left(\hat{\beta}_r - \beta\right)'\left(\hat{\beta}_r - \beta\right) \tag{51}$$

where $\hat{\beta}_r$ is any estimator considered in the r^{th} simulation.

The simulated scalar mean square errors of estimators are reported for different values of d, ρ, and n in Tables A1–A3 in Appendix. The general observation from the results of Tables A1–A3 is that the proposed estimator MAULLE performs well for large shrinkage values d compared to MLE, LLE, and AULLE in the scalar mean square error sense with respect to all the sample sizes $n = 15$, 50, and 100. It is further observed that if the multicollinearity is very high, the proposed estimator MAULLE is a very good alternative to MLE, LLE, and AULLE regardless of the values of n.

However, the performance of AULLE is considerably good for some selected values of d and ρ with respect to different sample sizes. That is, for $n = 15:$ when $\rho = 0.7, 0.8, 0.9$ & $d = 0.01, 0.1$; when $\rho = 0.99$ & $d = 0.01$. For $n = 50:$ when $\rho = 0.7$ & $0.2 \leq d \leq 0.6$; when $\rho = 0.8$ & $0.1 \leq d \leq 0.5$; when $\rho = 0.9$ & $0.01 \leq d \leq 0.3$; when $\rho = 0.99$ & $d = 0.01, 0.1$. For $n = 100:$ when $\rho = 0.7$ & $0.3 \leq d \leq 0.7$; when $\rho = 0.8$ & $0.2 \leq d \leq 0.6$; when $\rho = 0.9$ & $0.01 \leq d \leq 0.3$; when $\rho = 0.99$ & $d = 0.01, 0.1$. Further, MLE performs well compared to other estimators in the following cases: for $n = 50$, when $\rho = 0.7$ & $d = 0.01, 0.1$; when $\rho = 0.8$ & $d = 0.01$; for $n = 100$, when $\rho = 0.7$ & $0.01 \leq d \leq 0.2$; when $\rho = 0.8$ & $d = 0.01, 0.1$. Moreover, the range of d values for the superiority of the proposed estimator MAULLE with respect to different ρ and n is listed in Table A4 in Appendix.

5. A real data application

In this section, a real data application is given to check the performance of the proposed estimator with the existing estimators MLE, LLE, and AULLE. The data set is taken from the Statistics Sweden website (http://www.scb.se/) which consists the information of 100 municipalities of Sweden. The predictor variables considered in this example are Population ($x1$), Number unemployed people ($x2$), Number of newly constructed buildings ($x3$), and Number of bankrupt firms ($x4$). The response variable is Net population change (y) which takes the value 1 if there is an increase in the population and 0 otherwise.

All the pair-wise correlations of the explanatory variables $x1$, $x2$, $x3$, and $x4$ displayed in Table A5 are very high, and the condition number being a measure of multicollinearity is obtained as 188. This confirms the existence of severe multicollinearity in the data set. The SMSE values of MLE, LLE, AULLE, and MAULLE for some selected values of biasing parameter d in the range $0 < d < 1$ are given in the Table A6.

Results from the Table A6 reveal that the proposed estimator MAULLE outperforms the estimators MLE, LLE, and AULLE in the SMSE sense, with respect to the values of d in the range $0.5 \leq d \leq 0.99$ and AULLE performs well compared to other estimators when $0.01 \leq d \leq 0.4$.

6. Concluding remarks

In this paper, a new estimator called Modified almost unbiased logistic Liu-estimator (MAULLE) is proposed for logistic regression model when the multicollinearity problem exists. The superiority conditions for the proposed estimator with the existing estimators MLE, LLE, and AULLE are derived with respect to MSE and SMSE criteria. Further, from the real data application and the Monte Carlo simulation study we notice that for large biasing parameter d, the proposed estimator has smaller SMSE than MLE, LLE, and AULLE when a high multicollinearity exists among the explanatory variables.

Appendix

Table A1. The estimated MSE values for different d when $n = 15$.

		$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$	$\rho = 0.99$
$d = 0.01$	MLE	31.1712	48.2105	123.994	1156.8600
	LLE	15.3323	15.3216	15.7220	15.4536
	AULLE	**13.5468**	**13.5841**	**14.1253**	**13.7115**
	MAULLE	17.1466	17.0779	17.2939	17.1683
$d = 0.1$	MLE	31.1712	48.2105	123.9940	1156.8600
	LLE	16.0292	16.2333	17.4467	27.5016
	AULLE	**11.8959**	**12.6795**	**15.9513**	52.9457
	MAULLE	15.9552	15.9119	16.2128	**16.3816**
$d = 0.2$	MLE	31.1712	48.2105	123.9940	1156.8600
	LLE	17.2927	18.0492	21.5932	62.7244
	AULLE	**12.2224**	14.6768	25.0116	158.9460
	MAULLE	14.2370	**14.3056**	**14.9983**	**20.1172**
$d = 0.3$	MLE	31.1712	48.2105	123.9940	1156.8600
	LLE	19.0711	20.7103	28.0872	120.9330
	AULLE	14.1511	18.8479	39.0128	307.9250
	MAULLE	**12.6296**	**13.0494**	**15.1298**	**39.1105**
$d = 0.4$	MLE	31.1712	48.2105	123.9940	1156.8600
	LLE	21.3642	24.2164	36.9288	202.1270
	AULLE	17.0669	24.2894	55.7022	479.0890
	MAULLE	**11.6625**	**12.8446**	**18.1119**	**85.6926**
$d = 0.5$	MLE	31.1712	48.2105	123.9940	1156.8600
	LLE	24.1722	28.5677	48.1178	306.3060
	AULLE	20.4366	30.2186	73.1278	654.4170
	MAULLE	**11.7707**	**14.2700**	**25.2097**	**170.4530**
$d = 0.6$	MLE	31.1712	48.2105	123.9940	1156.8600
	LLE	27.4951	33.7641	61.6543	433.4710
	AULLE	23.8088	35.9731	89.6377	818.6600
	MAULLE	**13.2339**	**17.6861**	**37.1857**	**299.6360**

(*continued*)

Table A1. Continued.

		$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$	$\rho = 0.99$
$d = 0.7$	MLE	31.1712	48.2105	123.9940	1156.8600
	LLE	31.3328	39.8057	77.5383	583.6220
	AULLE	26.8145	41.0108	103.8800	959.3430
	MAULLE	**16.1372**	**23.1727**	**54.1228**	**473.3590**
$d = 0.8$	MLE	31.1712	48.2105	123.9940	1156.8600
	LLE	35.6853	46.6924	95.7699	756.7580
	AULLE	29.1665	44.9102	114.8040	1066.7600
	MAULLE	**20.3522**	**30.4974**	**75.3317**	**684.6650**
$d = 0.9$	MLE	31.1712	48.2105	123.9940	1156.8600
	LLE	40.5527	54.4242	116.3490	952.8790
	AULLE	30.6598	47.3704	121.6590	1133.9900
	MAULLE	**25.5393**	**39.1161**	**99.3447**	**919.3880**
$d = 0.99$	MLE	31.1712	48.2105	123.9940	1156.8600
	LLE	45.3735	62.1054	136.8780	1149.0400
	AULLE	31.1661	48.2020	123.9710	1156.6300
	MAULLE	**30.6063**	**47.3080**	**121.5750**	**1133.6900**

Table A2. The estimated MSE values for different d when $n = 50$.

		$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$	$\rho = 0.99$
$d = 0.01$	MLE	**4.7849**	**6.0705**	11.1276	97.4527
	LLE	8.9990	19.2242	9.5888	9.2170
	AULLE	5.6867	6.1118	**6.6473**	**5.5622**
	MAULLE	12.288	12.4509	12.7379	12.6760
$d = 0.1$	MLE	**4.7849**	6.0705	11.1276	97.4527
	LLE	9.2124	9.4978	10.0214	10.7044
	AULLE	4.9289	**5.4374**	**6.3221**	**9.2240**
	MAULLE	10.6922	10.8696	11.1584	11.0167
$d = 0.2$	MLE	4.7849	6.0705	11.1276	97.4527
	LLE	9.4815	9.8466	10.6201	14.0781
	AULLE	**4.4239**	**5.0434**	**6.4598**	18.0393
	MAULLE	8.9712	9.1760	9.5032	**9.7288**
$d = 0.3$	MLE	4.7849	6.0705	11.1276	97.4527
	LLE	9.7842	10.2424	11.3429	19.2632
	AULLE	**4.1878**	**4.9282**	**6.9656**	30.0977
	MAULLE	7.4076	7.6593	8.1095	**9.9672**
$d = 0.4$	MLE	4.7849	6.0705	11.1276	97.4527
	LLE	10.1204	10.6853	12.1899	26.2599
	AULLE	**4.1420**	**5.0046**	7.6952	43.7942
	MAULLE	6.0869	6.4110	**7.1092**	**12.7429**
$d = 0.5$	MLE	4.7849	6.0705	11.1276	97.4527
	LLE	10.4902	11.1752	13.1611	35.0682
	AULLE	**4.2184**	**5.1971**	8.5239	57.7376
	MAULLE	5.0699	5.4955	**6.5938**	**18.8738**
$d = 0.6$	MLE	4.7849	6.0705	11.1276	97.4527
	LLE	10.8936	11.7121	14.2564	45.6879
	AULLE	**4.3595**	5.4418	9.346	70.7510
	MAULLE	4.3907	**4.9472**	**6.6081**	**28.8064**
$d = 0.7$	MLE	4.7849	6.0705	11.1276	97.4527
	LLE	11.3305	12.2961	15.4758	58.1192
	AULLE	4.5181	5.6864	10.0750	81.8712
	MAULLE	**4.0554**	**4.7703**	**7.1469**	**42.4967**
$d = 0.8$	MLE	4.7849	6.0705	11.1276	97.4527
	LLE	11.801	12.9271	16.8195	72.3621
	AULLE	4.6576	5.8901	10.6438	90.3493
	MAULLE	**4.0433**	**4.9384**	**8.1547**	**59.3492**
$d = 0.9$	MLE	4.7849	6.0705	11.1276	97.4527
	LLE	12.3051	13.6051	18.2872	88.4164
	AULLE	4.7518	6.0239	11.0043	95.6501
	MAULLE	**4.3089**	**5.3977**	**9.5293**	**78.2158**
$d = 0.99$	MLE	4.7849	6.0705	11.1276	97.4527
	LLE	12.7874	14.2556	19.7144	104.414
	AULLE	4.7846	6.0700	11.1264	97.4346
	MAULLE	**4.7301**	**5.9962**	**10.9625**	**95.5678**

Table A3. The estimated MSE values for different d when $n = 100$.

		$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$	$\rho = 0.99$
$d = 0.01$	MLE	**2.4787**	**3.3107**	5.9266	26.2401
	LLE	6.4477	6.6175	6.2184	5.4998
	AULLE	3.3451	3.7899	**3.9522**	**3.1190**
	MAULLE	9.1262	9.1911	8.4145	7.6452
$d = 0.1$	MLE	**2.4787**	**3.3107**	5.9266	26.2401
	LLE	6.5528	6.7668	6.4862	6.1784
	AULLE	2.9155	3.4092	**3.9435**	**4.7124**
	MAULLE	7.8050	7.9115	7.2675	6.5020
$d = 0.2$	MLE	**2.4787**	3.3107	5.9266	26.2401
	LLE	6.6801	6.9483	6.8266	7.3200
	AULLE	2.6033	**3.1535**	**4.0823**	7.3202
	MAULLE	6.4410	6.5987	6.1584	**5.5979**
$d = 0.3$	MLE	2.4787	3.3107	5.9266	26.2401
	LLE	6.8184	7.1459	7.2122	8.8695
	AULLE	**2.4252**	**3.0316**	**4.3271**	10.4640
	MAULLE	5.2316	5.4475	5.2677	**5.2922**
$d = 0.4$	MLE	2.4748	3.3107	5.9266	26.2401
	LLE	6.9677	7.3599	7.6429	10.8271
	AULLE	**2.3445**	**3.0046**	4.6321	13.8277
	MAULLE	4.2120	4.4945	**4.6282**	**5.7805**
$d = 0.5$	MLE	2.4787	3.3107	5.9266	26.2401
	LLE	7.1280	7.5902	8.1187	13.1927
	AULLE	**2.3298**	**3.0387**	4.9575	17.1376
	MAULLE	3.4050	3.7630	**4.2568**	**7.1975**
$d = 0.6$	MLE	2.4787	3.3107	5.9266	26.2401
	LLE	7.2994	7.8367	8.6397	15.9662
	AULLE	**2.3542**	**3.1055**	5.2697	20.1616
	MAULLE	2.8207	3.2628	**4.1550**	**9.5919**
$d = 0.7$	MLE	2.4787	3.3107	5.9266	26.2401
	LLE	7.4818	8.0995	9.2058	19.1478
	AULLE	**2.3959**	3.1817	5.5410	22.7104
	MAULLE	2.4562	**2.9900**	**4.3078**	**12.9094**
$d = 0.8$	MLE	2.4787	3.3107	5.9266	26.2401
	LLE	7.6752	8.3786	9.8171	22.7374
	AULLE	2.4379	3.2490	5.7501	24.6362
	MAULLE	**2.2965**	**2.9278**	**4.6854**	**16.9861**
$d = 0.9$	MLE	2.4787	3.3107	5.9266	26.2401
	LLE	7.8797	8.6739	10.4735	26.7349
	AULLE	2.4679	3.2946	5.8817	25.8339
	MAULLE	**2.3158**	**3.0476**	**5.2437**	**21.5513**
$d = 0.99$	MLE	2.4787	3.3107	5.9266	26.2401
	LLE	8.0731	8.9536	11.1028	30.6816
	AULLE	2.4786	3.3105	5.9262	26.2361
	MAULLE	**2.4572**	**3.2792**	**5.8545**	**25.7777**

Table A4. Range of d values for superiority of MAULLE for different ρ and n.

	$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$	$\rho = 0.99$
$n = 15$	$0.3 \le d \le 0.99$	$0.2 \le d \le 0.99$	$0.2 \le d \le 0.99$	$0.1 \le d \le 0.99$
$n = 50$	$0.7 \le d \le 0.99$	$0.6 \le d \le 0.99$	$0.4 \le d \le 0.99$	$0.2 \le d \le 0.99$
$n = 100$	$0.8 \le d \le 0.99$	$0.7 \le d \le 0.99$	$0.5 \le d \le 0.99$	$0.2 \le d \le 0.99$

Table A5. The correlation matrix of the explanatory variables.

	$x1$	$x2$	$x3$	$x4$
$x1$	1.000	0.998	0.971	0.970
$x2$	0.998	1.000	0.960	0.958
$x3$	0.971	0.960	1.000	0.987
$x4$	0.970	0.958	0.987	1.000

Table A6. The SMSE values of estimators for the real world application data.

	MLE	LLE	AULLE	MAULLE
$d = 0.01$	0.000945756	0.000945754	**0.000944163**	0.000944164
$d = 0.10$	0.000945756	0.000945754	**0.000944310**	0.000944324
$d = 0.20$	0.000945756	0.000945755	**0.000944453**	0.000944464
$d = 0.30$	0.000945756	0.000945755	**0.000944616**	0.000944624
$d = 0.40$	0.000945756	0.000945755	**0.000944779**	0.000944784
$d = 0.50$	0.000945756	0.000945755	0.000944962	**0.000944945**
$d = 0.60$	0.000945756	0.000945755	0.000945126	**0.000945106**
$d = 0.70$	0.000945756	0.000945755	0.000945289	**0.000945268**
$d = 0.80$	0.000945756	0.000945755	0.000945452	**0.000945431**
$d = 0.90$	0.000945756	0.000945756	0.000945615	**0.000945593**
$d = 0.99$	0.000945756	0.000945756	0.000945762	**0.000945739**

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