Accelerating Convergence Rate of Linear Iteration Schemes Based on Projection Method for Three-stage Gauss Method

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Introduction: Consider an initial value problem for stiff system of $n(\geq 1)$ ordinary differential equations $x' = f(x(t)),\ x(t_0) = x_0,\ f:\mathbb{R}^n \to \mathbb{R}^n$. An s-stage implicit Runge-Kutta method computes an approximation x_{r+1} to the solution $x(t_{r+1})$ at discrete point $t_{r+1} = t_r + h$ by $x_{r+1} = x_r + h\sum_{i=1}^{s} b_i f(y_i)$ where the internal approximations $y_1, y_2, \ldots, y_s$ satisfy sn equations

$$y_i = x_r + h\sum_{j=1}^{s} a_{ij} f(y_j),\ i = 1,2,\ldots,s,$$

where $A = [a_{ij}]$ is the real coefficient matrix and $b = (b_1, b_2, \ldots, b_s)^T$ is the column vector of the Runge-Kutta method. Let $Y = y_1 \oplus y_2 \oplus \cdots \oplus y_s \in \mathbb{R}^{sn}$ and let $F(Y) = f(y_1) \oplus f(y_2) \oplus \cdots \oplus f(y_s) \in \mathbb{R}^{sn}$. Then the above equations in $y_1, y_2, \ldots, y_s$ written by $D(Y) = 0$, where D is the approximation defect defined by $D(Y) = e \otimes x_r + h(A \otimes I_n)F(Y)$, where $e = (1,1,\ldots,1)^T$ and $A \otimes I_n$ is the tensor product of the matrix A with $n \times n$ identity matrix. The non-linear system of equations $D(Y) = 0$ was solved by various linear iteration schemes proposed by various authors. In order to accelerate the convergence rate of the proposed linear iteration schemes Vigneswaran (2015) proposed a class of s-step nonlinear iteration scheme based on projection method and obtained the result for 2-stage Gauss method. Kajanthan and Vigneswaran (2017) obtained the result for 3-stage Gauss method by transforming the coefficient matrix and the iteration matrix to a block diagonal matrix. This work extends the Vigneswaran(2015) approach to 3-stage Gauss method.

Materials and Methods: The s-step scheme is given by

$$Y^m = Y^1,$$

$$\mu_i^m = \frac{\left[(Q\otimes I_n)(I_{sn} - hA\otimes J)E_i^m\right]^H (Q\otimes I_n)D(Y^{(i)})}{\left[(Q\otimes I_n)(I_{sn} - hA\otimes J)E_i^m\right]^H \left[(Q\otimes I_n)(I_{sn} - hA\otimes J)E_i^m\right]},$$

$$Y^{(i+1)} = Y^{(i)} + \mu_i^m E_i^m,\quad i = 1,2,3,\ldots,$$

$$Y^{m+1} = Y^{(s+1)},\quad m = 1,2,3,\ldots,$$

where μ_i^m is a scalar, $E_i^m = O \oplus O \oplus \cdots \oplus \varepsilon_i^m \oplus O \oplus \cdots \oplus O$, O the zero vector. In this scheme, sequence of numerical solutions is updated after each sub-step is completed. That is $Y^{(i)} = y_1^{m+1} \oplus y_2^{m+1} \oplus \cdots \oplus y_{i-1}^{m+1} \oplus y_i^m \oplus y_{i+1}^m \oplus \cdots \oplus y_s^m$ for $i = 1,2,3,\ldots,s$. The efficiency of this scheme was examined when it is applied to the linear scalar problem $y' = qy,\ q \in \mathbb{C}$ with rapid convergence required for all $z = hq$ in the left half complex plane, where h is a step size, and obtained the iteration matrix of this scheme. The non-singular matrix Q should be chosen to minimize the maximum of the spectral radius of the iteration matrix over the left half complex plane.

Results and Discussion: For 3-stage Gauss method, upper bound for the spectral radius of the iteration matrix was obtained in the left half complex plane. Numbers of numerical experiments were carried out in order to evaluate the efficiency of the proposed scheme here. For this, we consider an already proposed linear iteration scheme by Cooper and Butcher(1983) and accelerate its convergence rate using new proposed scheme. We compare the rate of convergence of the linear iteration scheme with its accelerated convergence rate by the new scheme for the same problem. Some stiff problems are considered for this comparison.

Conclusion: Numerical result shows that, for 3-stage Gauss method proposed scheme accelerates the convergence rate of the linear iteration scheme that we consider for the comparison in this work. It will be possible to apply the proposed class of non-linear scheme to accelerate the rate of convergence of other linear iteration schemes.

References :

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