Accelerating the Rate of Convergence of a More General Linear Iteration Scheme by using the Projection Method for Implicit Runge-Kutta Methods

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Abstract. Various iteration schemes are proposed by various authors to solve non-linear equations arising in the implementation of implicit Runge-Kutta methods. In this paper, a class of general non-linear scheme based on projection method is proposed to accelerate the rate of convergence of already proposed more general linear iteration scheme. Theoretical results are established in order to improve the rate of convergence of linear iteration schemes for two, three and four stage Gauss method. To confirm the results established in this paper, some numerical experiments are carried out.

Keywords - Gauss method, Implementation, Projection method, Rate of convergence, Stiff systems

1. INTRODUCTION

Consider an initial value problem for stiff system of ordinary differential equations

$$x' = f(x(t)), \quad x(t_0) = x_0, \quad f : \mathcal{R}^n \to \mathcal{R}^n. \qquad (1)$$

An s-stage implicit Runge-Kutta method computes an approximation x_{r+1} to the solution $x(t_{r+1})$ at discrete point $t_{r+1} = t_r + h$ by $x_{r+1} = x_r + h\sum_{i=1}^{s} b_i f(y_i)$ where the internal approximations $y_1, y_2, \ldots, y_s$ satisfy sn equations

$$y_i = x_r + h\sum_{j=1}^{s} a_{ij} f(y_j), \; i = 1,2,\ldots,s, \qquad (2)$$

and $A = [a_{ij}]$ is the real coefficient matrix of the Runge-Kutta method. Let $Y = y_1 \oplus y_2 \oplus \cdots \oplus y_s \in \mathcal{R}^{sn}$ and let $F(Y) = f(y_1) \oplus f(y_2) \oplus \cdots \oplus f(y_s) \in \mathcal{R}^{sn}$. Then the equations (2) written by where D is the approximation defect defined by D(y)=0, wher D is the approximation defect definced by

$$D(Y) = e \otimes x_r - Y + h(A \otimes I_n)F(Y), \qquad (3)$$

where $e = (1,1,\ldots,1)^T$ and $A \otimes I_n$ and $A \otimes I_n$ is the tensor product of the matrix A with identity matrix and, in general $A \otimes B = [a_{ij}B]$. This article deals with methods suitable for stiff systems so that the matrix is not strictly lower triangular. There are two general approaches proposed by several authors to solve the system. D (y) = O. In one approach, a modified Newton scheme is used. Let be the Jacobian of evaluated at some recent point updated infrequently. The modified Newton scheme evaluates $Y^1, Y^2, Y^3, \cdots$, to satisfy

$$(I_{sn} - hA \otimes J)(Y^m - Y^{m-1}) = D(Y^{m-1}), \quad m = 1,2,\ldots. \qquad (4)$$

In each step of this iteration, a set of sn linear equations has to be solved so that this scheme is still expensive. The other approach is to use schemes based directly on iterative procedures. In this type, several authors proposed several iteration schemes. A more general scheme was proposed by Cooper and Butcher [1]. This scheme sacrificing superlinear convergence for reduced linear algebra cost. They consider the scheme

$$[I_s \otimes (I_n - h\lambda J)]E^m = (BS^{-1} \otimes I_n)D(Y^{m-1}) + (L \otimes I_n)E^m,$$
$$Y^m = Y^{m-1} + (S \otimes I_n)E^m, \quad m = 1,2,\ldots, \qquad (5)$$

where B and S are real non-singular matrices and L is strictly lower triangular matrix of order s, and is a real constant. Cooper and Butcher [1] also showed that successive over-relaxation may be applied to improve the rate of convergence for scalar test problem. Cooper and Vigneswaran [2] proposed an efficient scheme where the $Y^m = y_1^m \oplus y_2^m \oplus \cdots \oplus y_s^m$ are obtained in sequence and the approximation defect is updated after each sub-step is completed. Only one vector transformation is needed for each full step. The rate of convergence of this scheme has been improved in [5 - 7]. Cooper and Vigneswaran [3] proposed another scheme, which is a generalization of the basic scheme (5), to obtain improved rate of convergence, by adding extra substeps. Further improvement in the rate of convergence of this scheme has been obtained in [8].

In this Paper, in order to accelerate the convergence rate of the proposed linear iteration schemes we propose a class of general non-linear scheme based on projection method.

2. A CLASS OF NON-LINEAR SCHEMES BASED ON PROJECTION METHOD

2.1 Projection Method for Linear system

Among the iterative methods for solving large linear algebraic systems of equations, those that have received the most attention have been the Jacobi and the Gauss-Seidel schemes, and their accelerated forms. But Householder [4] has dealt with another class of method which has been called projection method. Special cases include the method of steepest decent and the other gradient based schemes used to solve linear systems. The techniques have been used to accelerate convergence of iterative process for non-linear problems.

Consider solving the linear system Ax=b, where A is assumed to be a non-singular matrix. Let x_k represent any iterate and let $\delta_k = x - x_k$, $r_k = b - Ax_k$, represent the error and residual respectively, where x is the true solution. A method of projection is one in which at each step, the error

δ_k is resolved into two components, one of which is required to lie in a subspace selected at that step, and the other is δ_{k+1} which is required to be less than in some norm. The subspace is selected by choosing a matrix y_k whose columns are linearly independent and form a basis for the subspace. In practice y_k is generally a single vector y_k. That is, $\delta_{k+1} = \delta_k - y_k u_k$, where u_k is a vector (or scalar if y_k is a vector) to be selected at the step so that k^th^Householder shows that $\delta_{k+1} \leq \delta_k$ is minimized by choosing so that is the projection of onto the subspace spanned by the columns of with respect to G, where G is a positive definite matrix. This implies that is minimized when $Y_k^H = \overline{Y}_k^T$ where $y_K^H = Y_K^T$ is the Hermitian of y_k. Here $\|.\|$ is defined by $\|\delta_k\|^2 = \delta_K^H G \delta_k$.

2.2 A class of Non-Linear Scheme

The above idea is used to solve the non-linear system of equations D(Y)=0. Consider the iteration scheme of the form where is scalar and is a vector. Let In this new scheme, is chosen from the general linear iteration scheme. The scalar is chosen as is the projection onto with respect to a positive definite matrix where G is a non-singular matrix. Hence

$$Y^{m+1} = Y^m + \mu^m E^m, \quad m = 1,2,3,\ldots, \tag{6}$$

where $\mu^{m+1} = Y^m + \mu^m E^m$, is scalar and is a vector. Let In this new scheme, E^m is chosen from the general linear iteration scheme. The scalar μ^m is chosen as $\mu^m E^m$ is the projection Δ^m onto E^m with respect to a positive definite matrix $G^H G$, where G is a *sn x sn* non-singular matrix. Hence $\Delta^{m+1} = \Delta^m - \mu^m E^m$,

$$\mu^m = \frac{\left(GE^m\right)^H G\Delta^m}{\left(GE^m\right)^H \left(GE^m\right)}, \quad m = 1,2,3,\ldots. \tag{7}$$

Suppose that the sequence $Y^m \to Y$ as $m \to \infty$. If E^m is chosen so that $E^m \to 0$ gives $D\left(Y^m\right) \to 0$, it follows that $D(Y) = 0$. Here G and E^m have to be chosen so that the scheme can be efficiently implemented and performs well. In each step of the iteration (6) the scalar μ^m has to be calculated by using (7) but the numerator of μ^m contains Δ^m which is not known. To make the process feasible the matrix G may be chosen as $(Q \otimes I_n) D'\left(Y^m\right)$, where Q is a $s \times s$ non-singular matrix. Since $D\left(Y^m\right) = -D'\left(Y^m\right)\Delta^m + O\left(\|\Delta^m\|\right)^2$, $G\Delta^m$ may be approximated by $(Q \otimes I_n) D\left(Y^m\right)$. Since $F'\left(Y^m\right)$ is the block diagonal matrix and each diagonal block is the Jacobian of f at one of $y_1^m, y_2^m, \ldots, y_s^m$. Thus the evaluation of $D'\left(Y^m\right)$ requires more computation. To reduce this, the Jacobian is computed infrequently. Let J be the Jacobian evaluated at recent point x_p. Then $F'\left(x_p\right) = I_s \otimes J$ and $D'\left(Y^m\right) = -\left(I_{sn} - hA \otimes J\right)$. Hence from (7), He obtain

$$\mu^m = \frac{\left[(Q\otimes I_n)(I_{sn} - hA\otimes J)E^m\right]^H (Q\otimes I_n) D\left(Y^m\right)}{\left[(Q\otimes I_n)(I_{sn} - hA\otimes J)E^m\right]^H \left[(Q\otimes I_n)(I_{sn} - hA\otimes J)E^m\right]}, \tag{8}$$

where $E^m = E_1^m \oplus E_2^m \oplus \cdots \oplus E_s^m$ and $E_i^m = O \oplus O \oplus \cdots \oplus O \oplus \varepsilon_i^m \oplus O \oplus \cdots \oplus O$, O the zero vector.

The non-singular matrix Q and E^m have to be chosen so that the scheme performs well. The efficiency of this scheme examined when it is applied to the linear scalar problem $x' = qx$ with rapid convergence required for all $z = hq\,\mathbb{C}^- = \{z\,\mathbb{C} \mid \mathrm{Re}(z) \leq 0\}$. The iteration matrix is obtained in the following lemma.

Lemma 1. Let the scheme (6) be applied to the scalar problem $x' = qx$. Then there exists a strictly lower triangular matrix *L(z)* and a diagonal matrix *D(z)* such that $\Delta^{m+1} = -D^{-1}(z)\left[L(z) + L^H(z)\right]\Delta^m, \quad m = 1,2,\ldots.$

Proof: Since $D(Y) = 0$, for the test problem, $D\left(Y^{(i)}\right) = (I_s - zA)\left(Y - Y^{(i)}\right) = (I_s - zA)\Delta^{(i)}$. The scalar μ_i^m is given by $\mu_i^m = \frac{\left(E_i^m\right)^H (I_s - zA)^H Q^H Q D\left(Y^{(i)}\right)}{\left(E_i^m\right)^H (I_s - zA)^H Q^H Q (I_s - zA) E_i^m}$, where $E_i^m = \varepsilon_i^m e_i$, where ε_i^m is a scalar and $e_1, e_2, \ldots, e_s$ are the natural base vectors for $\mathcal{R}^s$. This implies

$$\mu_i^m = \frac{e_i^H (I_s - zA)^H Q^H Q (I_s - zA)\Delta\left(Y^{(i)}\right)}{e_i^H (I_s - zA)^H Q^H Q (I_s - zA) e_i}.$$

Let $l_{ij}(z) = e_i^H (I_s - zA)^H Q^H Q (I_s - zA) e_j$. It follows from the scheme (6)

$$\Delta^{(i+1)} = \Delta^{(i)} - \mu_i^m E_i^m, \quad i = 1,2,\ldots,s.$$

By equating the i^{th} component of both sides of the above equation, we obtain

$$e_i^H \Delta^{(i+1)} = e_i^H \Delta^{(i)} - e_i^H \mu_i^m E_i^m, \quad i = 1,2,\ldots,s.$$

This implies, for $i = 1,2,\ldots,s$,

$$\Delta_i^{(i+1)} = \Delta_i^{(i)} - \frac{1}{l_{ii}(z)}\left(e_i^H (I_s - zA)^H Q^H Q (I_s - zA)\right)\left(\Delta_1^m e_1 + \Delta_2^m e_2 + \cdots + \Delta_s^m e_s\right),$$

where $\Delta^m = \left(\Delta_1^m, \Delta_2^m, \ldots, \Delta_s^m\right)$ and $\Delta_i^{(i+1)} = \Delta_i^{(m+1)}$. Since $\overline{l}_{ij}(z) = l_{ji}(z)$, by rearranging the above equations, we can write the above equations in matrix form as $D(z)\Delta^{(m+1)} = -\left[L(z) + L^H(z)\right]\Delta^{(m)}$, where $L(z) = \left(l_{ij}(z)\right)$ is a strictly lower triangular matrix and $D(z) = \left(l_{ii}(z)\right)$ is a diagonal matrix. Then

$$\Delta^{m+1} = -D^{-1}(z)\left[L(z) + L^H(z)\right]\Delta^m, \quad m = 1,2,\ldots. \quad \blacksquare$$

The iteration matrix *M(z)* of the iteration (6) is given by

$$M(z) = -D^{-1}(z)\left[L(z) + L^H(z)\right]. \tag{9}$$

The elements of M(z) are independent of the choice of . Hence Q should be chosen to minimize the spectral radius of . This seems to be very difficult. We apply a different approach which is we impose spectral radius of M(z) to be zero for real z. The following theorem gives an upper bound for for the two stage Gauss method in the left half plane. The coefficient matrix of the two stage Gauss method is given by

$$A=\begin{bmatrix} a_1 & a_1-b_1 \\ a_1+b_1 & a_1 \end{bmatrix}, \tag{10}$$

where $a_1=\dfrac{1}{4}$ and $b_1=\dfrac{\sqrt{3}}{6}$.

Theorem 1. Consider the two-stage Gauss method with coefficient matrix given by (10) and S = I. Suppose that on the real axis z = x. Then there exists a non-singular matrix Q such that $Q^H Q=\begin{pmatrix} 1 & 0 \\ 0 & \dfrac{b_1-a_1}{b_1+a_1}\end{pmatrix}$ and $\rho[M(z)]\le\sqrt{1-\left(\dfrac{a_1}{b_1}\right)^2}$ for all $z\in\square^-$.

Proof: It follows from the **Lemma 1** that for two stage Gauss method the iteration matrix $M(z)=-D^{-1}(z)\left[L(z)+L^H(z)\right]$, where $L(z)=\begin{pmatrix} 0 & 0 \\ l_{21}(z) & 0\end{pmatrix}$, $D(z)=\begin{pmatrix} l_{11}(z) & 0 \\ 0 & l_{22}(z)\end{pmatrix}$.

This implies $M(z)=\begin{pmatrix} 0 & -\dfrac{l_{12}(z)}{l_{11}(z)} \\ -\dfrac{l_{21}(z)}{l_{22}(z)} & 0\end{pmatrix}$.

The spectral radius of the above iteration matrix is given by

$$\rho[M(z)]=\left|\sqrt{\frac{l_{12}(z)l_{21}(z)}{l_{11}(z)l_{22}(z)}}\right| \tag{11}$$

The elements $l_{ij}(z)'s$ are given in terms of the elements of $Q^H Q=\left[\alpha_{ij}\right]$ and the coefficient matrix of A by

$$\begin{aligned}
l_{11}(z)&=(1-\bar{z}a_1)\left[\alpha_{11}(1-za_1)-z\alpha_{12}(a_1+b_1)\right]\\
&\quad-\bar{z}(a_1+b_1)\left[\alpha_{21}(1-za_1)-z\alpha_{22}(a_1+b_1)\right],\\
l_{12}(z)&=(1-\bar{z}a_1)\left[-z\alpha_{11}(a_1-b_1)+\alpha_{12}(1-za_1)\right]\\
&\quad-\bar{z}(a_1+b_1)\left[-z\alpha_{21}(a_1-b_1)+\alpha_{22}(1-za_1)\right]\\
&=-z(a_1-b_1)\left[(1-\bar{z}a_1)\alpha_{11}-\bar{z}(a_1+b_1)\alpha_{21}\right]\\
&\quad+(1-za_1)\left[(1-\bar{z}a_1)\alpha_{12}-\bar{z}(a_1+b_1)\alpha_{22}\right],\\
l_{21}(z)&=-\bar{z}(a_1-b_1)\left[(1-za_1)\alpha_{11}-z(a_1+b_1)\alpha_{12}\right]\\
&\quad+(1-\bar{z}a_1)\left[(1-\bar{z}a_1)\alpha_{21}-z(a_1+b_1)\alpha_{22}\right]\\
&=\bar{l}_{12}(z),\\
l_{22}(z)&=-\bar{z}(a_1-b_1)\left[-z(a_1-b_1)\alpha_{11}+(1-za_1)\alpha_{12}\right]\\
&\quad+(1-\bar{z}a_1)\left[-z(a_1-b_1)\alpha_{21}+(1-za_1)\alpha_{22}\right].
\end{aligned} \tag{12}$$

We solve this set of equations by imposing the condition $\rho[M(z)]=0$ on the real axis $z=x$. That is from (11) we obtain $l_{12}(z)=0$. Hence the elements of $Q^H Q$ are obtained as

$\alpha_{12}=\alpha_{21}=0$, $\alpha_{11}=1$ and $\alpha_{22}=\dfrac{b_1-a_1}{b_1+a_1}$.

Then from (12), we obtain

$$\begin{aligned}
l_{11}&=1+z\bar{z}b_1^2-(z+\bar{z})a_1,\\
l_{12}&=(a_1-b_1)(\bar{z}-z),\\
l_{21}&=\bar{l}_{12},\\
l_{22}&=\frac{(a_1-b_1)\left(1+z\bar{z}b_1^2-(z+\bar{z})a_1\right)}{\left(1+z\bar{z}b_1^2-(z+\bar{z})a_1\right)^2}.
\end{aligned}$$

Since $\left(1+z\bar{z}b_1^2-(z+\bar{z})a_1\right)^2$ is positive for all $x,y\in\square$, then by the maximum modulus principle $\rho[M(z)]$ attains its maximum on the imaginary axis $z=iy$. This gives

$\rho[M(z)]=\sqrt{\dfrac{4\left(b_1^2-a_1^2\right)y^2}{\left(1+b_1^2y^2\right)^2}}\le\sqrt{\dfrac{b_1^2-a_1^2}{b_1^2}}$ on the imaginary axix $z=iy$. That is $\rho[M(z)]\le\sqrt{1-\left(\dfrac{a_1}{b_1}\right)^2}\ \square\ 0.5000$ for all $z\in\square^-$.

In this case we obtain

$$Q^H Q=\begin{pmatrix} 1 & 0 \\ 0 & \dfrac{b_1-a_1}{b_1+a_1}\end{pmatrix}. \tag{13}$$

In this approach, it is difficult to handle the 3-stage Gauss method and 4-stage Gauss method. We may transform the coefficient matrix and the iteration matrix to a block diagonal matrix. The result for s = 2 may be applied to other methods when s > 2. For each s- stage method of order 2s there is a real matrix S such that $S^{-1}AS=\bar{A}=A_1\oplus A_2\oplus\cdots\oplus A_r$ (14) real block diagonal matrix. The sub matrices are chosen to have the form

$$A_i=\begin{bmatrix} a_i & a_i-b_i \\ a_i+b_i & a_i\end{bmatrix},\quad i=1,2,\ldots,r, \tag{15}$$

with $b_i>a_i$ j=1,2....., ʳand except that, when s is odd $A^r=[a_r]$. Many iterative methods have coefficient matrices which may be transformed to real block diagonal matrices of the same form as (14). The iteration matrix M(*z*) can be written as a partition form corresponding to the partition of

$S^{-1}AS$:

$S^{-1}M(z)S=\bar{M}(z)=M_1(z)\oplus M_2(z)\oplus\cdots\oplus M_r(z)$.

Then the spectral radius is given by

$$\begin{aligned}
&\rho\left[\bar{M}(z)\right]=\max_{1\le i\le r}\rho\left[M_i(z)\right],\\
&M_i(z)=-D_i^{-1}(z)\left[L_i(z)+L_i^H(z)\right],\quad i=1,2,\ldots,r, \qquad (16)\\
&D(z)=D_1(z)\oplus D_2(z)\oplus\cdots\oplus D_r(z)\quad\text{and}\\
&L(z)=L_1(z)\oplus L_2(z)\oplus\cdots\oplus L_r(z)
\end{aligned}$$

corresponding to the partition of $S^{-1}AS$. When $s=3$ the method of order $2s$ has the matrix of coefficients

$$A=\begin{bmatrix}
\dfrac{5}{36} & \dfrac{2}{9}-\dfrac{\sqrt{15}}{15} & \dfrac{5}{36}-\dfrac{\sqrt{15}}{30}\\
\dfrac{5}{36}+\dfrac{\sqrt{15}}{24} & \dfrac{2}{9} & \dfrac{5}{36}-\dfrac{\sqrt{15}}{24}\\
\dfrac{5}{36}+\dfrac{\sqrt{15}}{30} & \dfrac{2}{9}+\dfrac{\sqrt{15}}{15} & \dfrac{5}{36}
\end{bmatrix}$$

and there is a matrix S such that

$$S^{-1}AS = \bar{A} = \begin{bmatrix} a_1 & a_1 - b_1 & 0 \\ a_1 + b_1 & a_1 & 0 \\ 0 & 0 & a_2 \end{bmatrix} = A_1 \oplus A_2,$$

where $a_1 \approx 0.142342788$, $b_1 \approx 0.196731007$, $a_2 \approx 0.215314423$ and a numerical calculation gives

$$S \approx \begin{bmatrix} -0.0455241821 & 0.0441943589 & 0.0721518521 \\ -0.140048242 & -0.139620426 & 0.118832579 \\ 1.0 & -0.244595668 & 1.0 \end{bmatrix}$$

where the columns are eigenvectors of $[a_1 I - A]^2$.

Let $D = D_1 \oplus D_2$ and $L = L_1 \oplus L_2$ so that the result of the **Theorem 1** may be applied using (16), we get

$$\rho[M_1(z)] \le \sqrt{1 - \left(\frac{a_1}{b_1}\right)^2} \approx 0.6903.$$

On the other hand, since $D_2 = [l_{33}]$ and $L_2 = [0]$, gives $M_2(z) = 0$ implies $\rho[M_2(z)] = 0$.

Then $\rho[\bar{M}(z)] = 0.6903$ and in this case we obtain

$$Q^H Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{b_1 - a_1}{b_1 + a_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (17)$$

Next, consider the four-stage Gauss method with matrix of coefficients $A = [a_{ij}]$ obtained by solving the sets of equations

$$\sum_{j=1}^{4} a_{ij} c_j^{r-1} = \frac{c_i^r}{r}, \quad r = 1,2,3,4, \quad \text{for each } i = 1,2,3,4, \text{ where}$$

c_1, c_2, c_3, c_4 are the zeros of $P_4(2x-1)$, the transformed legendre polynomial of degree 4. The elements of the transformed matrix

$$S^{-1}AS = \bar{A} = \begin{bmatrix} a_1 & a_1 + b_1 & 0 & 0 \\ a_1 + b_1 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & a_2 + b_2 \\ 0 & 0 & a_2 + b_2 & a_2 \end{bmatrix} = A_1 \oplus A_2,$$

where $a_1 \approx 0.091566240$, $a_2 \approx 0.158433760$, $b_1 \approx 0.147520224$, $b_2 \approx 0.165384116$ and

$$S \approx \begin{bmatrix} 0.063771667 & -0.054434907 & -0.231157907 & 0.013395896 \\ -0.027613999 & 0.161524607 & -0.083606572 & -0.040682019 \\ -0.784055901 & -0.290017081 & -0.859410259 & -0.266775537 \\ 1.0 & -1.164674610 & 1.0 & -1.364336800 \end{bmatrix}$$

where the columns are eigenvectors of $[a_1 I - A]^2$ and $[a_2 I - A]^2$. Again the result of the **Theorem 1** may be applied using (16), we obtain

$$\rho[M_1(z)] \le \sqrt{1 - \left(\frac{a_1}{b_1}\right)^2} \approx 0.7840,$$

$$\rho[M_2(z)] \le \sqrt{1 - \left(\frac{a_2}{b_2}\right)^2} \approx 0.2869,$$

where the matrices D and L are given by

$$L = L_1 \oplus L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ l_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & l_{43} & 0 \end{bmatrix}, \; D = D_1 \oplus D_2 = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ 0 & l_{22} & 0 & 0 \\ 0 & 0 & l_{33} & 0 \\ 0 & 0 & 0 & l_{44} \end{bmatrix}.$$

Then $\rho[\bar{M}(z)] = 0.7840$ and we obtain

$$Q^H Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{b_1 - a_1}{b_1 + a_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{b_2 - a_2}{b_2 + a_2} \end{pmatrix}. \qquad (18)$$

In the next section, obtained theoretical results were confirmed by numerical experiments

3. NUMERICAL RESULTS

A number of numerical experiments were carried out in order to evaluate the efficiency of the proposed class of general non-linear scheme. Results for three non-linear initial value problems are reported and compared with results obtained using the scheme described in Cooper and Butcher [1].

Problem 1 denotes the non-linear system

$$x_1' = -0.013x_1 + 1000x_1x_3, \qquad x_1(0) = 1,$$
$$x_2' = 2500x_2x_3, \qquad x_2(0) = 1,$$
$$x_3' = 0.013x_1 - 1000x_1x_3 - 2500x_2x_3, \qquad x_3(0) = 0,$$

where the eigenvalues of the Jacobian at the initial point are 0, -0.0093 and -3500.

Problem 2 is also the non-linear system

$$x_1' = -55x_1 + 65x_2 - x_1x_3, \qquad x_1(0) = 1,$$
$$x_2' = 0.0785(x_1 - x_2), \qquad x_2(0) = 1,$$
$$x_3' = 0.1x_1, \qquad x_3(0) = 0,$$

where, the eigenvalues of the Jacobian at the initial point are 0, -1.0 and -3.0 x 10^7

For each problem, a single step was carried out, in each method, using the Jacobian evaluated at the initial point. For each scheme tested, the initial iterate Y^0 is chosen as $Y^0 = e \otimes x$, here x is the true solution at the initial point.

Method 1 denotes the two-stage Gauss method implemented according to the basic scheme (5) with parameters given in Cooper and Butcher [1] with relaxation parameter ω=1.

Method 1* denotes the two-stage Gauss method but implemented using the non-linear scheme (6) proposed here with the matrix Q given by (13) and E^m chosen from the scheme (5).

Method 2 denotes the three-stage Gauss method implemented according to the basic scheme (5) with parameters given in Cooper and Butcher [1] with relaxation parameter ω=1.

Method 2* denotes the three-stage Gauss method but implemented using the non-linear scheme (6) proposed here with the matrix Q given by (17) and chosen from the scheme (5).

Method 3 denotes the four-stage Gauss method implemented according to the basic scheme (5) with parameters given in Cooper and Butcher [1] with relaxation parameter ω=1.

Method 3 denotes the four-stage Gauss method but implemented using the non-linear scheme (6) proposed here with the matrix Q given by (18) and E^m chosen from the scheme (5).

For each problem the quantities

$$e_m = \left\| Y^m - Y^{m-1} \right\|_\infty, \quad m = 1, 2, 3, \ldots,$$

are calculated. The values of $e_m \leq ToL = 10^{-9}$ are tabulated for each problem and method. Similar results are obtained for different values of TOL. The Results are given in the below table.

Table I
Values of m giving $e^m \leq 10^{-9}$ for Gauss method

Problems	Methods					
	1	1*	2	2*	3	3*
1	7	5	4	3	5	3
2	6	5	7	6	8	4
3	3	2	8	4	10	3

4. CONCLUSION

Numerical result shows that, the proposed class of general non- linear iteration scheme accelerates the convergence rate of the general linear iteration scheme proposed by Cooper and Butcher [1] for some stiff problems that has strong stiffness. It will be possible to apply the proposed class of general non-linear scheme to accelerate the rate of convergence of other linear iteration schemes.

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