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On the convergence of the accelerated Riccati iteration method

Prasanthan Rajasingam[a] and Jianhong Xu[b]

[a]Department of Mathematics & Statistics, University of Jaffna, Jaffna, Sri Lanka; [b]Department of Mathematics, Southern Illinois University Carbondale, Carbondale, Illinois, U.S.A.

ABSTRACT

In this paper, we establish results fully addressing two open problems proposed recently by I. Ivanov, see Nonlinear Analysis 69 (2008) 4012–4024, with respect to the convergence of the accelerated Riccati iteration method for solving the continuous coupled algebraic Riccati equation, or CCARE for short. These results confirm several desirable features of that method, including the monotonicity and boundedness of the sequences it produces, its capability of determining whether the CCARE has a solution, the extremal solutions it computes under certain circumstances, and its faster convergence than the regular Riccati iteration method.

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1. Introduction

In this paper, all matrices are real and square. The size of a matrix may not be specified if it is clear from the context. For the sake of brevity, a positive semidefinite matrix X is denoted by $X \succeq 0$. The term positive semidefinite, by convention, refers here only to the symmetric case, namely $X^T = X$. For symmetric matrices X and Y, $X \succeq Y$ means $X - Y \succeq 0$. Similarly, $X \preceq Y$ means $Y - X \succeq 0$. In addition, $\langle N \rangle$ stands for $\{1, 2, \ldots, N\}$.

The main problems we shall address in this paper concern the so-called continuous coupled algebraic Riccati equation, abbreviated as CCARE from now on. Specifically, let$A_i, S_i, Q_i \in \mathbb{R}^{n\times n}$, where $i \in \langle N \rangle$, and suppose that $S_i \succeq 0$ and $Q_i \succeq 0$ for all i, then the CCARE can be expressed in the form (Guo, 2013; Ivanov, 2008)

$$A_i^T X_i + X_i A_i - X_i S_i X_i + \sum_{j\in\langle N\rangle\setminus\{i\}} \delta_{i,j} X_j + Q_i = 0, \quad i \in \langle N \rangle, \tag{1}$$

where $\delta_{i,j} \geq 0$ for any $i \neq j$ and, moreover, $\sum_{j\in\langle N\rangle\setminus\{i\}} \delta_{i,j} > 0$ for each i. When there is no ambiguity, we shall denote by X_i, with $i \in \langle N \rangle$, a solution to the CCARE and call each X_i the ith component of the solution.

In particular, when $N = 1$, the CCARE reduces to the classical continuous algebraic Riccati equation, or CARE for short in the sequel, which can be written by removing the subscript i as

$$A^T X + XA - XSX + Q = 0, \tag{2}$$

where $S \succeq 0$ and $Q \succeq 0$. Throughout this paper, we shall always assume by default that $N \geq 2$ in (1). The CARE in (2), however, plays a critical role in dealing with the main problems here.

The CCARE in (1) arises originally from an optimal control problem on Markovian jump linear systems. For background material, see, for example, Costa, Fragoso, and Todorov (2013) and Mariton (1990). Due to its connection to the solution of the optimal control problem, the numerical computation of positive semidefinite solutions to the CCARE has drawn much attention in the literature, see Abou-Kandil, Freiling, and Jank (1994), Arnold and Laub (1984), Costa and do Val (2004), Damm and Hinrichsen (2001), Gajic and Borno (1995), Guo (2013), Ivanov (2007), Ivanov (2008), Kleinman (1968), Sandell (1974) and do Val, Geromel, and Costa (1999) and the references therein. Among these, the following two numerical methods are relevant here: one is the Riccati iteration method, whereas the other is the accelerated (or modified) Riccati iteration method.

We recall in passing the concepts of stabilizability and detectability. Let $A, S, Q \in \mathbb{R}^{n\times n}$. Then, (A, S) is called stabilizable if there exists matrix K such that $A - SK$ is stable, whereas (A, Q) is called detectable if (A^T, Q^T) is stabilizable. As a well-known result, such conditions guarantee the existence and uniqueness of a stabilising positive semidefinite solution to the CARE. This result will be stated formally in the next section.

The Riccati iteration method and its convergence are investigated in Costa and do Val (2004). This method can be

CONTACT Jianhong Xu jhxu@siu.edu Department of Mathematics, Southern Illinois University Carbondale,Carbondale, Illinois 62901, U.S.A.

formulated — see also (Ivanov, 2008) — as:

Algorithm 1.1: *For each $i \in \langle N\rangle$, choose the initial $X_i^{(0)} \succeq 0$ and set $\rho_i \geq 0$ such that $(A_i - \rho_i I, S_i)$ is stabilizable and $(A_i - \rho_i I, Q_i)$ is detectable. Next, for $k = 0, 1, 2, \ldots$, we iterate according to*

$$(A_i - \rho_i I)^T X_i^{(k+1)} + X_i^{(k+1)}(A_i - \rho_i I) - X_i^{(k+1)} S_i X_i^{(k+1)} + \sum_{j\in\langle N\rangle\setminus\{i\}} \delta_{i,j} X_j^{(k)} + Q_i + 2\rho_i X_i^{(k)} = 0, \quad i \in \langle N\rangle. \tag{3}$$

At each iteration, the above algorithm solves N CARE's, either in serial or in parallel if all $X_i^{(k)}$'s are available, which may be implemented easily in practice with Matlab's `care`. As mentioned in Costa and do Val (2004), however, the main advantage of Algorithm 1.1 is that the stabilizability and detectability conditions in this algorithm, i.e. in (3), can always be satisfied by choosing appropriate values of ρ_i's, and thus (3) computes unique sequences of positive semidefinite matrices $\{X_i^{(k)}\}$, $i \in \langle N\rangle$, even when the CCARE in (1) has no solution. Moreover, for each i, $\{X_i^{(k)}\}$ converges if and only if (1) has a solution, and it does so in a monotonically increasing fashion toward the minimal solution of (1), provided that $X_i^{(0)} = 0$ for all i; see Costa and do Val (2004) for more detail. Note that the latter feature here is especially attractive, since it means that the algorithm can also determine whether (1) has a solution or not, without resorting to conditions such as mean-square stabilizability and mean-square detectability (Guo, 2013).

The accelerated Riccati iteration method appears in Ivanov (2008, (20)) as an effort to improve upon Algorithm 1.1 via making use of updated $X_i^{(k+1)}$'s in (3) as soon as they become available. Intuitively, such a modification should speed up the convergence of Algorithm 1.1. Specifically, this accelerated algorithm can be summarised as:

Algorithm 1.2: *For each $i \in \langle N\rangle$, choose the initial $X_i^{(0)} \succeq 0$ and set $\rho_i \geq 0$ such that $(A_i - \rho_i I, S_i)$ is stabilizable and $(A_i - \rho_i I, Q_i)$ is detectable. Next, for $k = 0, 1, 2, \ldots$, we iterate according to*

$$(A_i - \rho_i I)^T X_i^{(k+1)} + X_i^{(k+1)}(A_i - \rho_i I) - X_i^{(k+1)} S_i X_i^{(k+1)} + \sum_{j=1}^{i-1} \delta_{i,j} X_j^{(k+1)} + \sum_{j=i+1}^{N} \delta_{i,j} X_j^{(k)} + Q_i + 2\rho_i X_i^{(k)} = 0, \quad i = 1, 2, \ldots, N. \tag{4}$$

Similar to the preceding one, at each iteration, the above accelerated algorithm solves N CARE's, but clearly only in a serial fashion — a potential trade-off between intrinsic parallelism and rate of convergence. Other shared features between the two algorithms are also expected here, such as the ease of implementation with available software and the existence and uniqueness of the sequences $\{X_i^{(k)}\}$, $i \in \langle N\rangle$, out of (4), consisting entirely of positive semidefinite matrices. Besides, when implemented with Matlab's `care` that is based on a generalised eigenproblem formulation (Arnold & Laub, 1984), Algorithms 1.2 and 1.1 share the same operation count for each iteration. In other words, Algorithm 1.2 will not increase the computational costs iteration-wise.

Algorithm 1.2, however, poses a number of interesting and crucial problems. Despite some favourable numerical evidence in Ivanov (2008), the following questions remain yet to be explored (Ivanov, 2008, p. 4021):

> Question 1: What conditions are needed for (4) to compute monotone, convergent sequences $\{X_i^{(k)}\}$, $i \in \langle N\rangle$?
> Question 2: Do such sequences converge faster in comparison to their counterparts from Algorithm 1.1?

The goals of this paper are to resolve these open problems that are vital to Algorithm 1.2.

2. Convergence of accelerated Riccati iteration method

Let us start with several necessary preparatory results on the solution of the CARE given by (2).

The first result here gives the necessary and sufficient conditions for the existence and uniqueness of a stabilising positive semidefinite solution to the CARE in terms of stabilizability and detectability.

Lemma 2.1 (Kučera, 1973, Theorem 5; also Bini, Lannazzo, & Meini, 2012, Theorem 2.21)**:** *The CARE in* (2) *has a unique positive semidefinite solution X such that $A-SX$ is stable, namely X is also stabilising, if and only if (A, S) is stabilizable and (A, Q) is detectable.*

The second result establishes an ordering for the solutions to (2) under a varying term Q. For convenience of application, we reformulate it based on its original form in Willems (1971).

Lemma 2.2 (Willems, 1971, Lemma 3, also Costa & do Val, 2004, Proposition 1)**:** *Suppose that $S \succeq 0$ and Q is symmetric. Let $X_1 \succeq 0$ be a solution of*

$$A^T X + XA - XSX + Q \preceq 0$$

such that $A - SX_1$ is stable and let $X_2 \succeq 0$ be a solution of

$$A^T X + XA - XSX + Q \succeq 0.$$

Then, $X_1 \succeq X_2$.

Finally, we cite below a result concerning detectability. Its original proof in Costa and do Val (2004) employs a rank argument, but it can also be shown alternatively using a well-known characterisation of detectability.

Lemma 2.3 (Costa & do Val, 2004, Proposition 2)**:** *Suppose that $Q \succeq 0$ and $\Delta Q \succeq 0$. Then, $(A, Q + \Delta Q)$ is detectable whenever so is (A, Q).*

Proof: By the Popov-Belevitch-Hautus tests, see Williams and Lawrence (2007, Theorem 8.5), (A, Q) is detectable if and only if there exists no (right) eigenvector u of A associated with eigenvalue λ with $\text{Re}\lambda \geq 0$ such that $Qu = 0$.

Let (A, Q) be detectable. Suppose now to the contrary that $(A, Q+\Delta Q)$ is not detectable. We denote by (λ, u), with $\mathrm{Re}\lambda \geq 0$, an eigenpair of A such that $(Q+\Delta Q)u = 0$. This leads to $u^*(Q+\Delta Q)u = u^*Qu + u^*\Delta Qu = 0$. In particular, we have $u^*Qu = 0$ and, consequently, $Qu = 0$, which is a contradiction to the detectability of (A, Q). ■

To facilitate the statement of our results, following Ivanov (2008), we define that for each $i \in \langle N\rangle$,

$$\mathcal{R}_i(X_1, X_2, \ldots, X_N) = A_i^T X_i + X_i A_i - X_i S_i X_i + \sum_{j\in\langle N\rangle\setminus\{i\}} \delta_{i,j} X_j + Q_i. \tag{5}$$

Accordingly, the CCARE in (1) can also be written as

$$\mathcal{R}_i(X_1, X_2, \ldots, X_N) = 0, \quad i \in \langle N\rangle.$$

We are now in a position to develop a number of results concerning the first question raised in Ivanov (2008), i.e. sufficient conditions so as to guarantee that the accelerated Riccati iteration method in Algorithm 1.2 computes unique monotonically increasing, bounded sequences of positive semidefinite matrices $\{X_i^{(k)}\}, i \in \langle N\rangle$.

Theorem 2.1: *Suppose that $\hat{X}_i \succeq 0$, $i \in \langle N\rangle$, are such that for each i,*

$$\mathcal{R}_i(\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_N) \preceq 0.$$

In addition, suppose that the initial positive semidefinite $X_i^{(0)}$'s in Algorithm 1.2 are such that

$$\mathcal{R}_i(X_1^{(0)}, X_2^{(0)}, \ldots, X_N^{(0)}) \succeq 0$$

and $X_i^{(0)} \preceq \hat{X}_i$ for all $i \in \langle N\rangle$. Moreover, for each i, let $\rho_i \geq 0$ be such that $(A_i - \rho_i I, S_i)$ is stabilizable, $(A_i - \rho_i I, Q_i)$ is detectable, and $A_i - \rho_i I - S_i\hat{X}_i$ is stable. Then,

(i) *Algorithm 1.2 computes unique sequences of positive semidefinite matrices $\{X_i^{(k+1)}\}$, where $k = 0, 1, 2, \ldots$ and $i \in \langle N\rangle$, such that for each i, $A_i - \rho_i I - S_i X_i^{(k+1)}$, where $k = 0, 1, 2, \ldots$, are all stable.*
(ii) *For each i, $X_i^{(k+1)} \succeq X_i^{(k)}$ for all $k = 0, 1, 2, \ldots$; that is, each $\{X_i^{(k)}\}$ is monotonically increasing.*
(iii) *For each i, $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}, \ldots, X_N^{(k)}) \succeq 0$ for all $k = 0, 1, 2, \ldots$.*
(iv) *For each i, $X_i^{(k)} \preceq \hat{X}_i$ for all $k = 0, 1, 2, \ldots$; that is, each $\{X_i^{(k)}\}$ is also bounded above.*

Proof: We proceed by way of induction on k and, for each k, induction on i as well.

Case $k = 0$. In this case, (iii) and (iv) are trivially true by assumption.

Let $i = 1$. From (4), we have

$$(A_1 - \rho_1 I)^T X_1^{(1)} + X_1^{(1)}(A_1 - \rho_1 I) - X_1^{(1)} S_1 X_1^{(1)} + Q_1 + \Delta Q_1 = 0, \tag{6}$$

where $\Delta Q_1 = \sum_{j=2}^N \delta_{1,j} X_j^{(0)} + 2\rho_1 X_1^{(0)} \succeq 0$. Since $(A_1 - \rho_1 I, S_1)$ is stabilizable and, following Lemma 2.3, $(A_1 - \rho_1 I, Q_1 + \Delta Q_1)$ is detectable, we know by Lemma 2.1 that (6) has a unique solution $X_1^{(1)} \succeq 0$ such that $A_1 - \rho_1 I - S_1 X_1^{(1)}$ is stable, and hence (i) holds at $k = 0$ and $i = 1$. In addition, using $\mathcal{R}_1(X_1^{(0)}, X_2^{(0)}, \ldots, X_N^{(0)}) \succeq 0$, we have

$$(A_1 - \rho_1 I)^T X_1^{(0)} + X_1^{(0)}(A_1 - \rho_1 I) - X_1^{(0)} S_1 X_1^{(0)} + Q_1 + \Delta Q_1 \succeq 0, \tag{7}$$

where ΔQ_1 is given as below (6). It follows from (6), (7), the stability of $A_1 - \rho_1 I - S_1 X_1^{(1)}$, and Lemma 2.2 that $X_1^{(1)} \succeq X_1^{(0)}$, i.e. (ii) holds as well at $k = 0$ and $i = 1$.

Suppose next that for some $2 \leq r \leq N$, (i) and (ii) are justified at $k = 0$ for all $i = 1, 2, \ldots, r-1$. On letting $i = r$ in (4), we obtain

$$(A_r - \rho_r I)^T X_r^{(1)} + X_r^{(1)}(A_r - \rho_r I) - X_r^{(1)} S_r X_r^{(1)} + Q_r + \Delta Q_r = 0, \tag{8}$$

where $\Delta Q_r = \sum_{j=1}^{r-1} \delta_{r,j} X_j^{(1)} + \sum_{j=r+1}^N \delta_{r,j} X_j^{(0)} + 2\rho_r X_r^{(0)} \succeq 0$. Since $(A_r - \rho_r I, S_r)$ is stabilizable while, from Lemma 2.3, $(A_r - \rho_r I, Q_r + \Delta Q_r)$ is detectable, (8) has a unique solution $X_r^{(1)} \succeq 0$ such that $A_r - \rho_r I - S_r X_r^{(1)}$ is stable according to Lemma 2.1, and hence (i) holds at $k = 0$. Finally, observe that $\mathcal{R}_r(X_1^{(0)}, X_2^{(0)}, \ldots, X_N^{(0)}) \succeq 0$ and $X_i^{(1)} \succeq X_i^{(0)}$, $i = 1, 2, \ldots, r-1$, yield

$$(A_r - \rho_r I)^T X_r^{(0)} + X_r^{(0)}(A_r - \rho_r I) - X_r^{(0)} S_r X_r^{(0)} + Q_r + \Delta Q_r \succeq 0, \tag{9}$$

where ΔQ_r is given under (8). Due to (8), (9), the stability of $A_r - \rho_r I - S_r X_r^{(1)}$, and Lemma 2.2, we see that $X_r^{(1)} \succeq X_r^{(0)}$, i.e. (ii) holds too at $k = 0$.

This concludes the proof of (i) through (iv) for the case $k = 0$.

Case $k > 0$. Suppose now that (i) through (iv) are true for some $k \geq 0$. We show here that they remain true at $k+1$.

First, by (4), and with (ii) and (iii) being true at k, it is clear that

$$\mathcal{R}_i(X_1^{(k+1)}, X_2^{(k+1)}, \ldots, X_N^{(k+1)}) \succeq 0, \quad i \in \langle N\rangle,$$

i.e. (iii) holds at $k+1$.

Next, for (i) and (ii), we start with $i = 1$. Using (4), we have

$$(A_1 - \rho_1 I)^T X_1^{(k+2)} + X_1^{(k+2)}(A_1 - \rho_1 I) - X_1^{(k+2)} S_1 X_1^{(k+2)} + Q_1 + \Delta\tilde{Q}_1 = 0, \tag{10}$$

where $\Delta\tilde{Q}_1 = \sum_{j=2}^N \delta_{1,j} X_j^{(k+1)} + 2\rho_1 X_1^{(k+1)} \succeq 0$. Since $(A_1 - \rho_1 I, S_1)$ is stabilizable and, via Lemma 2.3, $(A_1 - \rho_1 I, Q_1 + \Delta\tilde{Q}_1)$ is detectable, in view of Lemma 2.1, (10) has a unique solution $X_1^{(k+2)} \succeq 0$ with $A_1 - \rho_1 I - S_1 X_1^{(k+2)}$ being stable and, consequently, (i) is true at $k+1$ and $i = 1$. In addition, we find from $\mathcal{R}_1(X_1^{(k+1)}, X_2^{(k+1)}, \ldots, X_N^{(k+1)}) \succeq 0$ that

$$(A_1 - \rho_1 I)^T X_1^{(k+1)} + X_1^{(k+1)}(A_1 - \rho_1 I) - X_1^{(k+1)} S_1 X_1^{(k+1)} + Q_1 + \Delta\tilde{Q}_1 \succeq 0, \tag{11}$$

where $\Delta\tilde{Q}_1$ is given following (10). Because of the stability of $A_1 - \rho_1 I - S_1 X_1^{(k+2)}$ and Lemma 2.2, (10), and (11) imply $X_1^{(k+2)} \succeq X_1^{(k+1)}$, showing that (ii) also holds at $k+1$ and $i = 1$.

Suppose now that for some $2 \le r \le N$, both (i) and (ii) hold true for $i = 1, 2, \ldots, r-1$ at $k+1$. According to (4), we have

$$(A_r - \rho_r I)^T X_r^{(k+2)} + X_r^{(k+2)}(A_r - \rho_r I) - X_r^{(k+2)} S_r X_r^{(k+2)} + Q_r + \Delta\tilde{Q}_r = 0, \tag{12}$$

where $\Delta\tilde{Q}_r = \sum_{j=1}^{r-1} \delta_{r,j} X_j^{(k+2)} + \sum_{j=r+1}^{N} \delta_{r,j} X_j^{(k+1)} + 2\rho_r X_r^{(k+1)} \succeq 0$. Observe that, from Lemma 2.3, $(A_r - \rho_r I, Q_r + \Delta\tilde{Q}_r)$ is detectable. Besides, $(A_r - \rho_r I, S_r)$ is stabilizable. Hence, by Lemma 2.1, (12) has a unique solution $X_r^{(k+2)} \succeq 0$ such that $A_r - \rho_r I - S_r X_r^{(k+2)}$ is stable, implying that (i) is true at $k+1$. Finally, combining $\mathcal{R}_r(X_1^{(k+1)}, X_2^{(k+1)}, \ldots, X_N^{(k+1)}) \succeq 0$ and $X_i^{(k+2)} \succeq X_i^{(k+1)}$, $i = 1, 2, \ldots, r-1$, we arrive at

$$(A_r - \rho_r I)^T X_r^{(k+1)} + X_r^{(k+1)}(A_r - \rho_r I) - X_r^{(k+1)} S_r X_r^{(k+1)} + Q_r + \Delta\tilde{Q}_r \succeq 0, \tag{13}$$

where $\Delta\tilde{Q}_r$ is given under (12). By Lemma 2.2, (12), (13), and the stability of $A_r - \rho_r I - S_r X_r^{(k+2)}$ yield $X_r^{(k+2)} \succeq X_r^{(k+1)}$, i.e. (ii) holds at $k+1$.

It remains to show that (iv) is true at $k+1$, i.e. $X_i^{(k+1)} \preceq \hat{X}_i$, $i \in \langle N \rangle$. Again, we start with $i=1$. On one hand, because of $\mathcal{R}_1(\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_N) \preceq 0$, we have

$$(A_1 - \rho_1 I)^T \hat{X}_1 + \hat{X}_1(A_1 - \rho_1 I) - \hat{X}_1 S_1 \hat{X}_1 + Q_1 + \Delta\bar{Q}_1 \preceq 0, \tag{14}$$

where $\Delta\bar{Q}_1 = \sum_{j=2}^{N} \delta_{1,j} \hat{X}_j + 2\rho_1 \hat{X}_1 \succeq 0$. On the other hand, seeing (4) along with $X_i^{(k)} \preceq \hat{X}_i$, $i \in \langle N \rangle$, we obtain

$$(A_1 - \rho_1 I)^T X_1^{(k+1)} + X_1^{(k+1)}(A_1 - \rho_1 I) - X_1^{(k+1)} S_1 X_1^{(k+1)} + Q_1 + \Delta\bar{Q}_1 \succeq 0, \tag{15}$$

where $\Delta\bar{Q}_1$ is given as below (14). Since $A_1 - \rho_1 I - S_1 \hat{X}_1$ is stable, accordingly to Lemma 2.2, we get from (14) and (15) that $X_1^{(k+1)} \preceq \hat{X}_1$.

Suppose next that for some $2 \le r \le N$, $X_i^{(k+1)} \preceq \hat{X}_i$, $i = 1, 2, \ldots, r-1$. By $\mathcal{R}_r(\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_N) \preceq 0$, we get

$$(A_r - \rho_r I)^T \hat{X}_r + \hat{X}_r(A_r - \rho_r I) - \hat{X}_r S_r \hat{X}_r + Q_r + \Delta\bar{Q}_r \preceq 0, \tag{16}$$

where $\Delta\bar{Q}_r = \sum_{j=1}^{r-1} \delta_{r,j} \hat{X}_j + \sum_{j=r+1}^{N} \delta_{r,j} \hat{X}_j + 2\rho_r \hat{X}_r \succeq 0$. In the meantime, we use (4) together with $X_i^{(k+1)} \preceq \hat{X}_i$, where $i = 1, 2, \ldots, r-1$, and $X_i^{(k)} \preceq \hat{X}_i$, where $i = r+1, r+2, \ldots, N$, to derive

$$(A_r - \rho_r I)^T X_r^{(k+1)} + X_r^{(k+1)}(A_r - \rho_r I) - X_r^{(k+1)} S_r X_r^{(k+1)} + Q_r + \Delta\bar{Q}_r \succeq 0, \tag{17}$$

where $\Delta\bar{Q}_r$ is given next to (16). Finally, the stability of $A_r - \rho_r I - S_r \hat{X}_r$, (16), (17), and Lemma 2.2 lead to $X_r^{(k+1)} \preceq \hat{X}_r$. This shows that (iv) holds too at $k+1$.

The proof is now complete in its entirety. ∎

An immediate consequence of Theorem 2.1 goes as follows.

Corollary 2.1: *Under the same conditions as Theorem 2.1, Algorithm 1.2 computes unique sequences of positive semidefinite matrices $\{X_i^{(k)}\}$, with $i \in \langle N \rangle$, that converge to a positive semidefinite solution X_i, $i \in \langle N \rangle$, of the CCARE in (1), i.e. $\lim_{k\to\infty} X_i^{(k)} = X_i$ for each i.*

Proof: For each i, the convergence of $\{X_i^{(k)}\}$ is obvious — see, for example, Ivanov, Hasanov, and Minchev (2001) and Xu and Xiao (2013, Corollary 4.1) — and it does so toward some positive semidefinite X_i. Next, by pushing $k \to \infty$ in (4), we see that X_i, $i \in \langle N \rangle$, is indeed a solution to (1). ∎

Corollary 2.1 shows that, similar to the pure Riccati iteration method in Algorithm 1.1, the accelerated version here in Algorithm 1.2 can also determine whether the CCARE has a solution or not. To be specific, Algorithm 1.2 yields a positive semidefinite solution to the CCARE whenever it converges.

Furthermore, if $\hat{X}_i$'s in Theorem 2.1 happen to be a positive semidefinite solution to the CCARE in (1), then Algorithm 1.2 actually finds the minimal positive semidefinite solution to (1) as the next result demonstrates.

Corollary 2.2: *Let $X_i \succeq 0$, $i \in \langle N \rangle$, be a solution to (1). Suppose that the initial positive semidefinite $X_i^{(0)}$'s in Algorithm 1.2 are such that*

$$\mathcal{R}_i(X_1^{(0)}, X_2^{(0)}, \ldots, X_N^{(0)}) \succeq 0$$

and $X_i^{(0)} \preceq X_i$ for all $i \in \langle N \rangle$. Moreover, for each i, let $\rho_i \ge 0$ be such that $(A_i - \rho_i I, S_i)$ is stabilizable, $(A_i - \rho_i I, Q_i)$ is detectable, and $A_i - \rho_i I - S_i X_i$ is stable. Then, Algorithm 1.2 produces unique sequences of positive semidefinite matrices $\{X_i^{(k)}\}$, $i \in \langle N \rangle$, such that for each i, $\{X_i^{(k)}\}$ is monotonically increasing, bounded above by X_i, and converges to X_i^-, the ith component of the minimal positive semidefinite solution to the CCARE in(1).

Proof: It is clear that Corollary 2.2 assumes the same conditions as Theorem 2.1, except for $\hat{X}_i$'s being replaced with X_i's. By Corollary 2.1, we know that Algorithm 1.2 computes a positive semidefinite solution X_i^-, $i \in \langle N \rangle$, to (1). Besides, note that due to Theorem 2.1, $X_i^- \preceq X_i$ for all i whenever X_i, $i \in \langle N \rangle$, is a solution to (1), thus X_i^-, $i \in \langle N \rangle$, is the minimal positive semidefinite solution to (1). ∎

Clearly, Corollaries 2.1 and 2.2 also verify that Algorithm 1.2 converges if and only if (1) has a positive semidefinite solution, provided that the initial $X_i^{(0)}$'s are chosen as in Corollary 2.2. We point out that, in particular, those conditions on $X_i^{(0)}$'s are trivially satisfied when $X_i^{(0)} = 0$, $i \in \langle N \rangle$. In other words, this desirable feature of the Riccati iteration method for allowing a full determination of the existence of a positive semidefinite solution — see (Costa & do Val, 2004) — carries over to the accelerated Riccati iteration method here.

In light of Theorem 2.1 and Corollaries 2.1 and 2.2, we can formulate the following three parallel results, whose proofs are very similar and, therefore, are omitted.

Theorem 2.2: *Suppose that $\hat{X}_i \succeq 0$, $i \in \langle N\rangle$, are such that for each i,*

$$\mathcal{R}_i(\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_N) \succeq 0.$$

In addition, suppose that the initial positive semidefinite $X_i^{(0)}$'s in Algorithm 1.2 are such that

$$\mathcal{R}_i(X_1^{(0)}, X_2^{(0)}, \ldots, X_N^{(0)}) \preceq 0$$

and $X_i^{(0)} \succeq \hat{X}_i$ for all $i \in \langle N\rangle$. Moreover, for each i, let $\rho_i \geq 0$ be such that $(A_i - \rho_i I, S_i)$ is stabilizable, $(A_i - \rho_i I, Q_i)$ is detectable, and $A_i - \rho_i I - S_i X_i^{(0)}$ is stable. Then,

(i) *Algorithm 1.2 computes unique sequences of positive semidefinite matrices $\{X_i^{(k+1)}\}$, where $k = 0, 1, 2, \ldots$ and $i \in \langle N\rangle$, such that for each i, $A_i - \rho_i I - S_i X_i^{(k+1)}$, where $k = 0, 1, 2, \ldots$, are all stable.*
(ii) *For each i, $X_i^{(k+1)} \preceq X_i^{(k)}$ for all $k = 0, 1, 2, \ldots$; that is, each $\{X_i^{(k)}\}$ is monotonically decreasing.*
(iii) *For each i, $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}, \ldots, X_N^{(k)}) \preceq 0$ for all $k = 0, 1, 2, \ldots$.*
(iv) *For each i, $X_i^{(k)} \succeq \hat{X}_i$ for all $k = 0, 1, 2, \ldots$; that is, each $\{X_i^{(k)}\}$ is also bounded below.*

Corollary 2.3: *Under the same conditions as Theorem 2.2, Algorithm 1.2 computes unique sequences of positive semidefinite matrices $\{X_i^{(k)}\}$, with $i \in \langle N\rangle$, that converge to a positive semidefinite solution X_i, $i \in \langle N\rangle$, of the CCARE in (1), i.e. $\lim_{k\to\infty} X_i^{(k)} = X_i$ for each i.*

Corollary 2.4: *Let $X_i \succeq 0$, $i \in \langle N\rangle$, be a solution to (1). Suppose that the initial positive semidefinite $X_i^{(0)}$'s in Algorithm 1.2 are such that*

$$\mathcal{R}_i(X_1^{(0)}, X_2^{(0)}, \ldots, X_N^{(0)}) \preceq 0$$

and $X_i^{(0)} \succeq X_i$ for all $i \in \langle N\rangle$. Moreover, for each i, let $\rho_i \geq 0$ be such that $(A_i - \rho_i I, S_i)$ is stabilizable, $(A_i - \rho_i I, Q_i)$ is detectable, and $A_i - \rho_i I - S_i X_i^{(0)}$ is stable. Then, Algorithm 1.2 produces unique sequences of positive semidefinite matrices $\{X_i^{(k)}\}$, $i \in \langle N\rangle$, such that for each i, $\{X_i^{(k)}\}$ is monotonically decreasing, bounded below by X_i, and converges to X_i^+, the ith component of the maximal positive semidefinite solution to the CCARE in(1).

Corollaries 2.1 through 2.4, coupled with Theorems 2.1 and 2.2, serve as a rather complete answer to the first open problem in Ivanov (2008). Especially, these corollaries spell out not only the conditions for convergence in Algorithm 1.2 but also the particular extremal types of solution this algorithm converges to under certain circumstances. Moreover, these results can be regarded as new characterisations for the existence of extremal solutions to the CCARE, extending Ran and Vreugdenhil (1988, Theorem 2.1) which concerns the maximal solution to the CARE.

It is straightforward to see that, in fact, Algorithm 1.1 shares all of the preceding results on Algorithm 1.2. The proofs are very similar except that the inductive steps on *i* are no longer needed. For the sake of concision, we only state such results without proof in forms parallel to Theorems 2.1 and 2.2. In addition, for clarity, we denote the sequences from Algorithm 1.1 by $\{Y_i^{(k)}\}$'s here.

Theorem 2.3: *Suppose that $\hat{X}_i \succeq 0$, $i \in \langle N\rangle$, are such that for each i,*

$$\mathcal{R}_i(\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_N) \preceq 0.$$

In addition, suppose that the initial positive semidefinite $Y_i^{(0)}$'s in Algorithm 1.1 are such that

$$\mathcal{R}_i(Y_1^{(0)}, Y_2^{(0)}, \ldots, Y_N^{(0)}) \succeq 0$$

and $Y_i^{(0)} \preceq \hat{X}_i$ for all $i \in \langle N\rangle$. Moreover, for each i, let $\rho_i \geq 0$ be such that $(A_i - \rho_i I, S_i)$ is stabilizable, $(A_i - \rho_i I, Q_i)$ is detectable, and $A_i - \rho_i I - S_i \hat{X}_i$ is stable. Then,

(i) *Algorithm 1.1 computes unique sequences of positive semidefinite matrices $\{Y_i^{(k+1)}\}$, where $k = 0, 1, 2, \ldots$ and $i \in \langle N\rangle$, such that for each i, $A_i - \rho_i I - S_i Y_i^{(k+1)}$, where $k = 0, 1, 2, \ldots$, are all stable.*
(ii) *For each i, $Y_i^{(k+1)} \succeq Y_i^{(k)}$ for all $k = 0, 1, 2, \ldots$; that is, each $\{Y_i^{(k)}\}$ is monotonically increasing.*
(iii) *For each i, $\mathcal{R}_i(Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_N^{(k)}) \succeq 0$ for all $k = 0, 1, 2, \ldots$.*
(iv) *For each i, $Y_i^{(k)} \preceq \hat{X}_i$ for all $k = 0, 1, 2, \ldots$; that is, each $\{Y_i^{(k)}\}$ is also bounded above.*

Theorem 2.4: *Suppose that $\hat{X}_i \succeq 0$, $i \in \langle N\rangle$, are such that for each i,*

$$\mathcal{R}_i(\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_N) \succeq 0.$$

In addition, suppose that the initial positive semidefinite $Y_i^{(0)}$'s in Algorithm 1.1 are such that

$$\mathcal{R}_i(Y_1^{(0)}, Y_2^{(0)}, \ldots, Y_N^{(0)}) \preceq 0$$

and $Y_i^{(0)} \succeq \hat{X}_i$ for all $i \in \langle N\rangle$. Moreover, for each i, let $\rho_i \geq 0$ be such that $(A_i - \rho_i I, S_i)$ is stabilizable, $(A_i - \rho_i I, Q_i)$ is detectable, and $A_i - \rho_i I - S_i Y_i^{(0)}$ is stable. Then,

(i) *Algorithm 1.1 computes unique sequences of positive semidefinite matrices $\{Y_i^{(k+1)}\}$, where $k = 0, 1, 2, \ldots$ and $i \in \langle N\rangle$, such that for each i, $A_i - \rho_i I - S_i Y_i^{(k+1)}$, where $k = 0, 1, 2, \ldots$, are all stable.*
(ii) *For each i, $Y_i^{(k+1)} \preceq Y_i^{(k)}$ for all $k = 0, 1, 2, \ldots$; that is, each $\{Y_i^{(k)}\}$ is monotonically decreasing.*
(iii) *For each i, $\mathcal{R}_i(Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_N^{(k)}) \preceq 0$ for all $k = 0, 1, 2, \ldots$.*
(iv) *For each i, $Y_i^{(k)} \succeq \hat{X}_i$ for all $k = 0, 1, 2, \ldots$; that is, each $\{Y_i^{(k)}\}$ is also bounded below.*

Compared with the results in Costa and do Val (2004), Theorems 2.3 and 2.4 on Algorithm 1.1 are broader because, firstly, they allow nonzero initial $Y_i^{(0)}$'s and, secondly, they provide respective sufficient conditions for the resulting convergent sequences $\{Y_i^{(k)}\}$'s to be either monotonically increasing or monotonically decreasing. Consequently, conclusions on extremal solutions Algorithm 1.1 can compute follow easily from these theorems — in a way similar to Corollaries 2.2 and 2.4.

It should be pointed out here that being able to determine the extremal solutions to the CCARE is indeed a nice feature of Algorithms 1.1 and 1.2 with practical implications, since it is known (Rami & Ghaoui, 1996; do Val & Costa, 2005) that such solutions are useful in solving optimal control problems.

We comment that in Theorem 2.1, Corollary 2.2, and Theorem 2.3, as in Costa and do Val (2004) and Ivanov (2008), an easy choice of the initial $X_i^{(0)}$'s and $Y_i^{(0)}$'s is to set $X_i^{(0)} = Y_i^{(0)} = 0$ for any $i \in \langle N\rangle$. With this choice, all the conditions on $X_i^{(0)}$'s and $Y_i^{(0)}$'s in those results are trivially satisfied. On the other hand, when applying Theorem 2.2, Corollary 2.4, and Theorem 2.4, we may choose the initial $X_i^{(0)}$'s and $Y_i^{(0)}$'s to be some existing upper solution bounds for the CCARE. For results relevant to such bounds, see, for example, Czornik and Swierniak (2001), Davies, Shi, and Wiltshire (2008), Xu (2013) and Xu and Rajasingam (2016) and the references therein.

One of the advantages shared by Algorithms 1.1 and 1.2 is that the stabilizability and detectability requirements can always be met by appropriate values of ρ_i's. In Theorem 2.1, Corollary 2.2, and Theorem 2.3, however, the choice of ρ_i's is complicated by the stability requirement on $A_i - \rho_i I - S_i X_i$'s since, in practice, the solution X_i's is not available *a priori*. Although this issue might be alleviated by resorting to sufficiently large ρ_i values, we shall demonstrate later that, similar to Algorithm 1.1, unnecessarily large ρ_i values are usually not advisable for Algorithm 1.2.

Next, we move on to examining the other open problem in Ivanov (2008) regarding a comparison of the rate of convergence of the accelerated Riccati iteration method versus that of the Riccati iteration method. In this regard, we prove the following:

Theorem 2.5: *Under the same assumptions as in Theorems* 2.1 *and* 2.3 *with* $X_i^{(0)} = Y_i^{(0)}$ *for all* $i \in \langle N\rangle$, *on letting* $\{X_i^{(k)}\}$, $i \in \langle N\rangle$, *be the sequences computed with Algorithm* 1.2 *and* $\{Y_i^{(k)}\}$, $i \in \langle N\rangle$, *be the corresponding sequences computed with Algorithm* 1.1, *we have that for each i,* $X_i^{(k)} \succeq Y_i^{(k)}$, *where* $k = 0, 1, 2, \ldots$.

Proof: Again, we use induction on k and, given k, induction on i. The case $k = 0$ is trivial here.

Suppose now that at some $k \geq 0$,

$$X_i^{(k)} \succeq Y_i^{(k)}, \quad i \in \langle N\rangle. \tag{18}$$

Let us show that $X_i^{(k+1)} \succeq Y_i^{(k+1)}$, $i \in \langle N\rangle$.

From (4) and (18), we obtain

$$(A_1 - \rho_1 I)^T X_1^{(k+1)} + X_1^{(k+1)}(A_1 - \rho_1 I) - X_1^{(k+1)} S_1 X_1^{(k+1)} + \sum_{j=2}^{N} \delta_{1,j} Y_j^{(k)} + Q_1 + 2\rho_1 Y_1^{(k)} \preceq 0. \tag{19}$$

In the meantime, we see by setting $i = 1$ in (3) that

$$(A_1 - \rho_1 I)^T Y_1^{(k+1)} + Y_1^{(k+1)}(A_1 - \rho_1 I) - Y_1^{(k+1)} S_1 Y_1^{(k+1)} + \sum_{j=2}^{N} \delta_{1,j} Y_j^{(k)} + Q_1 + 2\rho_1 Y_1^{(k)} = 0. \tag{20}$$

Using Lemma 2.2 and noting the stability of $A_1 - \rho_1 I - S_1 X_1^{(k+1)}$ from part (i) of Theorem 2.1, (19) and (20) lead to $X_1^{(k+1)} \succeq Y_1^{(k+1)}$.

Next, suppose that for some $2 \leq r \leq N$,

$$X_i^{(k+1)} \succeq Y_i^{(k+1)}, \quad i = 1, 2, \ldots, r-1. \tag{21}$$

It follows from (4), (18), and (21) that

$$(A_r - \rho_r I)^T X_r^{(k+1)} + X_r^{(k+1)}(A_r - \rho_r I) - X_r^{(k+1)} S_r X_r^{(k+1)} + \sum_{j=1}^{r-1} \delta_{r,j} Y_j^{(k+1)} + \sum_{j=r+1}^{N} \delta_{r,j} Y_j^{(k)} + Q_r + 2\rho_r Y_r^{(k)} \preceq 0. \tag{22}$$

Moreover, we see from (3) and from the monotonicity of $\{Y_i^{(k)}\}$ established in Theorem 2.3 that

$$(A_r - \rho_r I)^T Y_r^{(k+1)} + Y_r^{(k+1)}(A_r - \rho_r I) - Y_r^{(k+1)} S_r Y_r^{(k+1)} + \sum_{j=1}^{r-1} \delta_{r,j} Y_j^{(k+1)} + \sum_{j=r+1}^{N} \delta_{r,j} Y_j^{(k)} + Q_r + 2\rho_r Y_r^{(k)} \succeq 0. \tag{23}$$

Using Lemma 2.2 again and noting the stability of $A_r - \rho_r I - S_r X_r^{(k+1)}$ from part (i) of Theorem 2.1, (22) and (23) yield $X_r^{(k+1)} \succeq Y_r^{(k+1)}$, which implies that $X_i^{(k+1)} \succeq Y_i^{(k+1)}$, $i \in \langle N\rangle$.

This finishes the proof. ■

Since with the assumptions of Theorems 2.1 and 2.3, both Algorithms 1.1 and 1.2 compute unique increasing sequences of positive semidefinite matrices, Theorem 2.5 indicates that in this case, Algorithm 1.2 tends to converge faster than Algorithm 1.1.

In the same spirit as Theorem 2.5, we can state below a parallel conclusion, whose proof is obvious and thus is omitted.

Theorem 2.6: *Under the same assumptions as in Theorems* 2.2 *and* 2.4 *with* $X_i^{(0)} = Y_i^{(0)}$ *for all* $i \in \langle N\rangle$, *on letting* $\{X_i^{(k)}\}$, $i \in \langle N\rangle$, *be the sequences computed with Algorithm* 1.2 *and* $\{Y_i^{(k)}\}$, $i \in \langle N\rangle$, *be the corresponding sequences computed with Algorithm* 1.1, *we have that for each i,* $X_i^{(k)} \preceq Y_i^{(k)}$, *where* $k = 0, 1, 2, \ldots$.

The above Theorems 2.5 and 2.6, together, provide an answer to the second open problem in Ivanov (2008).

Returning to the issue regarding the choice of ρ_i's, similar to Algorithm 1.1 — see (Costa & do Val, 2004, Remark 2), we now illustrate that these parameters should be picked in such a way that they are as small as possible. Numerical examples in this regard can be found in Ivanov (2008).

For the ease of statement, we first modify (4) to that for $k = 0, 1, 2, \ldots$,

$$[A_i - (\rho_i + \Delta\rho_i)I]^T Y_i^{(k+1)} + Y_i^{(k+1)}[A_i - (\rho_i + \Delta\rho_i)I]$$
$$- Y_i^{(k+1)} S_i Y_i^{(k+1)} + \sum_{j=1}^{i-1} \delta_{i,j} Y_j^{(k+1)} + \sum_{j=i+1}^{N} \delta_{i,j} Y_j^{(k)} + Q_i$$
$$+ 2(\rho_i + \Delta\rho_i) Y_i^{(k)} = 0, \tag{24}$$

where $i \in \langle N \rangle$ and $\Delta\rho_i \geq 0$ for all i; namely we consider a setting in which each ρ_i in (4) is augmented by $\Delta\rho_i$. Note that, here and in the sequel, we denote the sequences generated by (24) as $\{Y_i^{(k)}\}$ so as to differentiate them from $\{X_i^{(k)}\}$ generated by (4). Clearly, the stabilizability and detectability conditions in Theorem 2.1, when it holds, still apply to (24).

Theorem 2.7: *Under the same assumptions as Theorem* 2.1 *with $X_i^{(0)} = Y_i^{(0)}$ for all $i \in \langle N \rangle$, let $\{X_i^{(k)}\}$, $i \in \langle N \rangle$, be the sequences computed from* (4) *and let $\{Y_i^{(k)}\}$, $i \in \langle N \rangle$, be the corresponding sequences computed from*(24), *then we have that $X_i^{(1)} \succeq Y_i^{(1)}$, $i \in \langle N \rangle$.*

Proof: Observe first that (24) satisfies all the conditions stated in Theorem 2.1. Hence,$\{Y_i^{(k)}\}$'s are uniquely determined by (24) and have all the properties in Theorem 2.1.

Let us prove the conclusion by induction on i. At $i=1$, we obtain from $X_1^{(0)} = Y_1^{(0)} \preceq Y_1^{(1)}$ and (24) that

$$(A_1 - \rho_1 I)^T Y_1^{(1)} + Y_1^{(1)}(A_1 - \rho_1 I) - Y_1^{(1)} S_1 Y_1^{(1)}$$
$$+ \sum_{j=2}^{N} \delta_{1,j} X_j^{(0)} + Q_1 + 2\rho_1 X_1^{(0)} = 2\Delta\rho_1 (Y_1^{(1)} - Y_1^{(0)}) \succeq 0. \tag{25}$$

Comparing (25) and (4) with $i=1$, and noting the stability of $A_1 - \rho_1 I - S_1 X_1^{(1)}$, we see $X_1^{(1)} \succeq Y_1^{(1)}$ by Lemma 2.2.

Next, suppose that there exists some $2 \leq r \leq N$ such that $X_i^{(1)} \succeq Y_i^{(1)}$, $i = 1, 2, \ldots, r-1$. This, together with $X_i^{(0)} = Y_i^{(0)} \preceq Y_i^{(1)}$ for all i and (24), yield

$$(A_r - \rho_r I)^T Y_r^{(1)} + Y_r^{(1)}(A_r - \rho_r I) - Y_r^{(1)} S_r Y_r^{(1)} + \sum_{i=1}^{r-1} \delta_{r,j} X_j^{(1)}$$
$$+ \sum_{j=r+1}^{N} \delta_{r,j} X_j^{(0)} + Q_r + 2\rho_r X_r^{(0)} \succeq 2\Delta\rho_r (Y_r^{(1)} - Y_r^{(0)}) \succeq 0. \tag{26}$$

Comparing (26) to (4) with $i=r$, and in presence of the stability of $A_r - \rho_r I - S_r X_r^{(1)}$, we conclude using Lemma 2.2 that $X_r^{(1)} \succeq Y_r^{(1)}$.

Thus, $X_i^{(1)} \succeq Y_i^{(1)}$ for all $i \in \langle N \rangle$. ■

In Theorem 2.7 above, for uniform satisfaction of the conditions in Theorem 2.1 on both (4) and (24), we follow Costa and do Val (2004, Remark 2) to perform only a 'single step' analysis. This analysis, however, extends essentially to the scenario$X_i^{(k+1)} \succeq Y_i^{(k+1)}$, $i \in \langle N \rangle$, whenever $X_i^{(k)} = Y_i^{(k)}$ for all i. Accordingly, this result justifies that, in general, the larger ρ_i's are, the slower the convergence (4), i.e. Algorithm 1.2, tends to exhibit.

Finally, in the same vein as Theorem 2.7, we formulate here without proof its counterpart assuming the conditions in Theorem 2.2.

Theorem 2.8: *Under the same assumptions as Theorem* 2.2 *with $X_i^{(0)} = Y_i^{(0)}$ for all $i \in \langle N \rangle$, let $\{X_i^{(k)}\}$, $i \in \langle N \rangle$, be the sequences computed from* (4) *and let $\{Y_i^{(k)}\}$, $i \in \langle N \rangle$, be the corresponding sequences computed from*(24), *then we have that $X_i^{(1)} \preceq Y_i^{(1)}$, $i \in \langle N \rangle$.*

3. Numerical results

To illustrate our main conclusions in the preceding section, we present here relevant numerical results on one example. In accordance with the primary goals of this work, our numerical experiment has been carried out only with the Riccati iteration method, i.e. Algorithm 1.1, and the accelerated Riccati iteration method, i.e. Algorithm 1.2. For numerical results comparing these methods with other existing methods, we refer the reader to Ivanov (2008). Moreover, in view of our results, the example we provide here features distinct minimal and maximal positive semidefinite solutions.

Example 3.1: Let $n=N=2$. Let $A_1 = \left[\begin{smallmatrix} 1 & -2 \\ 0 & -1 \end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix} 1 & -1 \\ 0 & -3 \end{smallmatrix}\right]$, $S_1 = B_1 B_1^T$, where $B_1 = \left[\begin{smallmatrix} 5 \\ -5 \end{smallmatrix}\right]$, $S_2 = B_2 B_2^T$, where $B_2 = \left[\begin{smallmatrix} 6 \\ 3 \end{smallmatrix}\right]$, $\delta_{1,2} = 2$, $\delta_{2,1} = 3$, $Q_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right]$, and $Q_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 3/2 \end{smallmatrix}\right]$. Then, the CCARE as in (1) has the minimal positive semidefinite solution

$$X_1^- = \begin{bmatrix} 0.00000000 & 0.00000000 \\ 0.00000000 & 0.28204532 \end{bmatrix},$$
$$X_2^- = \begin{bmatrix} 0.00000000 & 0.00000000 \\ 0.00000000 & 0.27641488 \end{bmatrix}$$

and the maximal positive semidefinite solution

$$X_1^+ = \begin{bmatrix} 0.50718185 & 0.24899225 \\ 0.24899225 & 0.45594482 \end{bmatrix},$$
$$X_2^+ = \begin{bmatrix} 0.32609148 & -0.16073063 \\ -0.16073063 & 0.48929635 \end{bmatrix}.$$

In our numerical experiment, the stopping criterion is set as

$$\max_{i \in \langle 2 \rangle} \|X_i^{(k)} - X_i^{(k-1)}\|_F < \text{tol} = 10^{-8},$$

where $\|\cdot\|_F$ stands for the Frobenius norm. Upon the termination of either algorithm at the mth iteration, the residual is calculated by

$$\max_{i \in \langle 2 \rangle} \|\mathcal{R}_i(X_1^{(m)}, X_2^{(m)})\|_F,$$

where $\mathcal{R}_i$ is given in (5). In addition, for each i, we denote the largest eigenvalue of $X_i^{(k)}$ by$\lambda_1(X_i^{(k)})$, the smallest eigenvalue of $X_i^{(k)}$ by $\lambda_2(X_i^{(k)})$, and the spectrum of $X_i^{(k)}$ by $\sigma(X_i^{(k)})$, i.e.

Table 1. This table shows, as ρ_i values vary, the numbers of iterations and residuals from Algorithms 1.1 and 1.2 when computing X_i^-.

$\rho_1 = \rho_2$	Algorithm 1.1		Algorithm 1.2	
	# of iterations	residual	# of iterations	residual
1.5	17	3.92×10^{-8}	14	4.25×10^{-8}
1.1	16	1.91×10^{-8}	13	1.43×10^{-8}
1.01	16	1.20×10^{-8}	12	3.48×10^{-8}

Table 2. This table shows, as ρ_i values vary, the numbers of iterations and residuals from Algorithms 1.1 and 1.2 in the case of computing X_i^+.

$\rho_1 = \rho_2$	Algorithm 1.1		Algorithm 1.2	
	# of iterations	residual	# of iterations	residual
1.5	42	3.31×10^{-8}	38	2.67×10^{-8}
1.1	36	2.86×10^{-8}	32	1.38×10^{-8}
1.01	35	2.25×10^{-8}	30	1.75×10^{-8}

$\sigma(X_i^{(k)}) = \{\lambda_1(X_i^{(k)}), \lambda_2(X_i^{(k)})\}$. These quantities are used in the illustrations.

To compute X_i^-, we choose $X_1^{(0)} = X_2^{(0)} = 0$. It is not difficult to verify that the conditions of Corollary 2.2 are all satisfied for any $\rho_i > 1$, $i = 1,2$. For $\rho_1 = \rho_2 = 1.01$, Algorithm 1.1 converges to X_i^- in 16 iterations, while Algorithm 1.2 does so in 12 iterations, as shown in the left panel in Figure 1. In the meantime, the right panel of Figure 1 displays the spectra of $X_i^{(k)}$ computed from Algorithm 1.2, which shows that for each i, $\{X_i^{(k)}\}$ is monotonically increasing as confirmed by Corollary 2.2.

With varying ρ_i values, we summarise in Table 1 the resulting numbers of iterations and residuals for computing X_i^- by Algorithms 1.1 and 1.2. It points to that, as suggested by Theorem 2.5, Algorithm 1.2 converges faster than Algorithm 1.1. It also shows the speed-up in Algorithm 1.2 along with decreasing values of ρ_i, see Theorem 2.7.

Next, to compute X_i^+, we choose $X_1^{(0)} = X_2^{(0)} = 3I$. It is quite straightforward to verify that the conditions in Corollary 2.4 are all satisfied for all $\rho_i > 1$, $i = 1,2$. Given $\rho_1 = \rho_2 = 1.01$, Algorithm 1.1 converges to X_i^+ in 35 iterations, while Algorithm 1.2 does so in 30 iterations, as illustrated by the left panel in Figure 2. In the right panel of Figure 2, the spectra of $X_i^{(k)}$ obtained from Algorithm 1.2 are plotted, showing that for each i, $\{X_i^{(k)}\}$ is monotonically decreasing.

With the same decreasing values of ρ_i as in Table 1, we provide in Table 2 evidence as indicated by Theorem 2.8 of a speed-up in Algorithm 1.2 for computing X_i^+. As a comparison, the corresponding numerical results from Algorithm 1.1 are given in Table 2 as well. From these results, we also see that, as indicated by Theorem 2.6, Algorithm 1.2 tends to converge faster than Algorithm 1.1 too when it comes to computing X_i^+.

4. Concluding remarks

The focus of this paper is on the two open problems raised in Ivanov (2008) concerning the monotone convergence of the accelerated Riccati iteration method as well as its rate of convergence in comparison with the pure Riccati iteration method. Our results aim mainly to settle these problems. In the process, we also broaden and strengthen some existing results in Costa and do Val (2004).

A unique and quite useful feature of the Riccati iteration method and its accelerated version is their adoption of parameters ρ_i's, which leads to easy satisfaction of the stabilizability and detectability conditions. In view of such parameters, we may call these methods 'shifted' Riccati iteration methods as versus the 'unshifted' Riccati iteration methods in do Val, Geromel, and Costa (1999). Even though we have provided some guidelines for choosing ρ_i's, the numerical determination of the best ρ_i's in practice is an interesting topic for future research.

The idea of utilising the updated $X_i^{(k+1)}$'s in the regular Riccati iteration method can be regarded as an extension to similar works on the accelerated Lyapunov iteration method (Guo, 2013; do Val, Geromel, and Costa, 1999). These, besides Ivanov (2008), have also motivated our development in this paper of theoretical results on the pure and accelerated Riccati iteration methods.

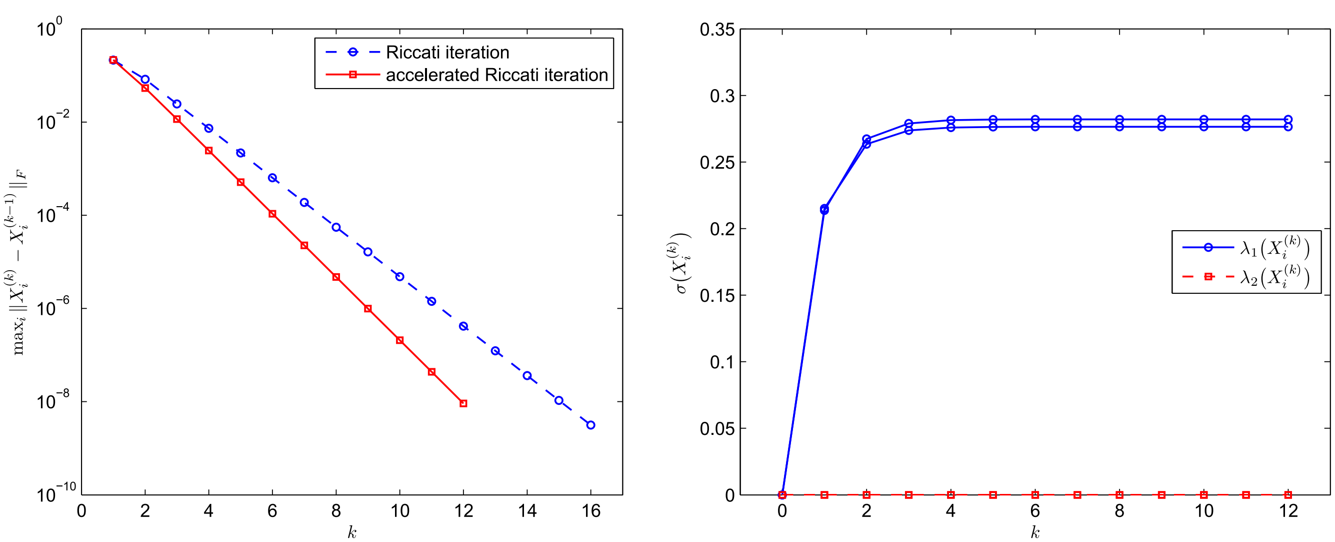

Figure 1. The case of computing X_i^- when $\rho_1 = \rho_2 = 1.01$: The left panel shows $\max_{i\in\langle 2\rangle} \|X_i^{(k)} - X_i^{(k-1)}\|_F$ from Algorithms 1.1 and 1.2, whereas the right panel illustrates the monotonic increasingness of the sequences $\{X_i^{(k)}\}$ obtained from Algorithm 1.2. Note that in this case, $\lambda_2(X_i^{(k)}) = 0$ for all i and all k.

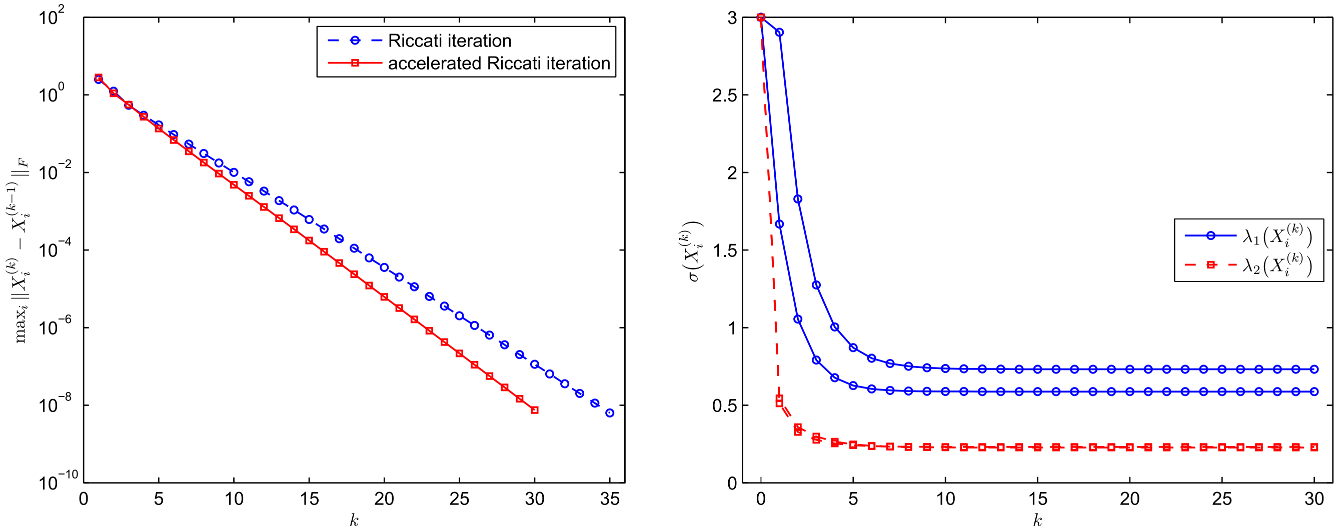

Figure 2. The case of computing X_i^+ when $\rho_1 = \rho_2 = 1.01$: The left panel shows $\max_{i\in\langle 2\rangle} \|X_i^{(k)} - X_i^{(k-1)}\|_F$ from Algorithms 1.1 and 1.2, whereas the right panel shows the monotonic decreasingness of the sequences $\{X_i^{(k)}\}$ obtained from Algorithm 1.2.

Throughout this paper, we assume exact arithmetic in analysing the two methods here. From a practical perspective, however, the stability and sensitivity analyses on these methods appear to be an interesting topic for future research too.

Another interesting topic for further investigation is a theoretical analysis comparing the performance of the two methods here with that of other existing numerical methods for solving the CCARE. In Ivanov (2008), for example, we can find only numerical results concerning the performances of the methods under consideration here, Newton's method, together with the Lyapunov and the accelerated Lyapunov iteration methods. Nevertheless, several theoretical results on the performances of the 'unshifted' Riccati methods and the Lyapunov iteration methods are presented in do Val, Geromel, and Costa (1999). We expect, therefore, that parallel results in this regard may also be developed to include the 'shifted' Riccati iteration methods.

Recalling the remark following Theorem 2.4, upper solution bounds play an important role in numerical computations on the CCARE. In fact, lower solution bounds are equally important. In Corollary 2.2, for example, $X_i^{(0)}$'s are indeed lower solution bounds. We feel that much work is still needed on simpler, tighter, and more easily applicable upper and lower solution bounds for the CCARE along with their applications in solving the CCARE numerically.

Last but not least, the original framework in Costa and do Val (2004) is more general in that it recasts the CCARE as one of the special cases from a so-called perturbed algebraic Riccati equation, abbreviated as PARE, involving a monotonically increasing positive semidefinite operator. It is one more important problem for us to explore as to whether the results here can be extended, with some splittings of that operator, to more effectively handle the general PARE.

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