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Fredholm operators and essential S-spectrum in the quaternionic setting

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For bounded right linear operators, in a right quaternionic Hilbert space with a left multiplication defined on it, we study the approximate S -point spectrum. In the same Hilbert space, then we study the Fredholm operators and the Fredholm index. In particular, we prove the invariance of the Fredholm index under small norm operator and compact operator perturbations. Finally, in association with the Fredholm operators, we develop the theory of essential S -spectrum. We also characterize the S -spectrum in terms of the essential S -spectrum and Fredholm operators. In the sequel, we study left and right S -spectra as needed for the development of the theory presented in this note. *Published by AIP Publishing.* <https://doi.org/10.1063/1.5040017>

I. INTRODUCTION

In the complex theory, Fredholm operators play an important role in the investigations of various classes of singular integral equations, in the theory of perturbations of Hermitian operators by Hermitian and non-Hermitian operators, and in obtaining *a priori* estimate in determining properties of certain differential operators.^{24,15,8,14,18} The Fredholm alternative theorem is used to derive an adjoint equation to the linear stability equations in fluid dynamics and useful in scattering of a 1-D particle on a small potential barrier; see Ref. 15 and the many references therein.

In the complex case, studies of Fredholm theory and perturbation results are of great importance in the description of the essential spectrum. In several applications, such as the essential spectrum of the Schrödinger equations, essential spectrum of the perturbed Hamiltonians, in general, essential spectrum of the differential operators, essential spectrum of the transport operator (transport of the neutron, photons, molecules in gas, etc.), in particular, the time dependent transport equations arise in a number of different applications in Biology, Chemistry, and Physics, and information about it is important.^{3,14–16,12,24} Furthermore, near the essential spectrum, numerical calculations of eigenvalues become difficult. Hence they have to be treated analytically.¹⁵ There are several distinct definitions of the essential spectrum. However they all coincide for the self-adjoint operators in Hilbert space.^{14,3} In this note, we only consider the essential spectrum associated with Fredholm operators in the quaternionic setting.

In the complex setting, in a Hilbert space \mathfrak{H} , a bounded linear operator, A , is not invertible if it is not bounded below (the same is true in the quaternion setting; see Theorem 3.7). The set of approximate eigenvalues which are $\lambda \in \mathbb{C}$ such that $A - \lambda \mathbb{I}_{\mathfrak{H}}$ is not bounded below, where $\mathbb{I}_{\mathfrak{H}}$ is the identity operator on \mathfrak{H} , equivalently, the set of $\lambda \in \mathbb{C}$ for which there is a sequence of unit vectors ϕ_1, ϕ_2, \dots such that $\lim_{n \rightarrow \infty} \|A\phi_n - \lambda\phi_n\| = 0$. The set of approximate eigenvalues is known as the approximate spectrum. In the quaternionic setting, let $V_{\mathbb{H}}^R$ be a separable right Hilbert space, A be a bounded right linear operator, and $R_q(A) = A^2 - 2\text{Re}(q)A + |q|^2 \mathbb{I}_{V_{\mathbb{H}}^R}$, with $q \in \mathbb{H}$, the set of all quaternions, be the pseudo-resolvent operator, and the set of right eigenvalues of $R_q(A)$ coincide with the point S -spectrum (see Proposition 4.5 in Ref. 13). In this regard, it will be appropriate to define and study

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the quaternionic approximate S-point spectrum as the quaternions for which $R_q(A)$ is not bounded below.

Due to the non-commutativity, in the quaternionic case, there are three types of Hilbert spaces: left, right, and two-sided, depending on how vectors are multiplied by scalars. This fact can entail several problems. For example, when a Hilbert space \mathcal{H} is one-sided (either left or right), the set of linear operators acting on it does not have a linear structure. Moreover, in a one-sided quaternionic Hilbert space, given a linear operator A and a quaternion $q \in \mathbb{H}$, in general, we have that $(qA)^\dagger \neq \bar{q}A^\dagger$ (see Ref. 20 for details). These restrictions can severely prevent the generalization to the quaternionic case of results valid in the complex setting. Even though most of the linear spaces are one-sided, it is possible to introduce a notion of multiplication on both sides by fixing an arbitrary Hilbert basis of \mathcal{H} . This fact allows us to have a linear structure on the set of linear operators, which is a minimal requirement to develop a full theory. Thus, the framework of this paper, in part, is a right quaternionic Hilbert space equipped with a left multiplication, introduced by fixing a Hilbert basis.

As far as we know, the Fredholm operator theory and the essential S-spectrum and the approximate S-point spectrum have not been studied in the quaternionic setting yet. In this regard, in this note, we investigate the quaternionic S-point spectrum, Fredholm operators, and associated S-essential spectrum for a bounded right linear operator in a right quaternionic separable Hilbert space. Since the pseudo-resolvent operator, $R_q(A)$, has real coefficients, the left multiplication defined in a right quaternionic Hilbert space plays a little role. Even the non-commutativity of quaternions does not play an essential role. Even though the S-approximate point spectrum, the S-essential spectrum, and the Fredholm operators are structurally different from its complex counterparts, the results we obtain and their proofs are somewhat similar to those in the corresponding complex theory.

The article is organized as follows. In Sec. II, we introduce the set of quaternions and quaternionic Hilbert spaces and their bases, as needed for the development of this article, which may not be familiar to a broad range of audiences. In Sec. III, we define and investigate, as needed, right linear operators and their properties. We have given proofs for some results which are not available in the literature. In Sec. III A, we define a basis dependent left multiplication in a right quaternionic Hilbert space. In Sec. III B, we deal with the right S-spectrum, left S-spectrum, S-spectrum, and its major partitions. In Sec. IV, we provide a systematic study of compact operators which has not yet been done in the literature. In Sec. V, we study the approximate S-point spectrum, $\sigma_{ap}^S(A)$, of a bounded right linear operator, A , in a right quaternionic Hilbert space. In particular, we prove that $\sigma_{ap}^S(A)$ is a non-empty closed subset of \mathbb{H} and the S-spectrum is the union of the $\sigma_{ap}^S(A)$ and the continuous S-spectrum. In Sec. VI, we study the Fredholm operators and its index for a bounded right linear operator. In particular, we prove the invariance of the Fredholm index under small norm operator and compact operator perturbations. In this section, since a quaternionic multiple of an operator is not involved, the proofs of these results are almost verbatim copies of the complex ones. In Sec. VII, we study the essential S-spectrum as the S-spectrum of the quotient map image of a bounded right linear operator on the quaternionic version of the Calkin algebra and then characterize the S-essential spectrum in terms of Fredholm operators. We also establish the so-called Atkinsons theorem, prove the invariance of the essential S-spectrum under compact perturbations, and give a characterization to the S-spectrum in terms of Fredholm operators and its index (see Proposition 7.18). Section VIII ends the manuscript with a conclusion.

II. MATHEMATICAL PRELIMINARIES

In order to make the paper self-contained, we recall some facts about quaternions which may not be well known. For details, we refer the reader to Refs. 1, 13, and 23.

A. Quaternions

Let \mathbb{H} denote the field of all quaternions and \mathbb{H}^* denote the group (under quaternionic multiplication) of all invertible quaternions. A general quaternion can be written as

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \quad q_0, q_1, q_2, q_3 \in \mathbb{R},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the three quaternionic imaginary units, satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$. The quaternionic conjugate of q is

$$\bar{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3,$$

while $|q| = (q\bar{q})^{1/2}$ denotes the usual norm of the quaternion q . If q is the non-zero element, it has inverse $q^{-1} = \frac{\bar{q}}{|q|^2}$. Finally, the set

$$\mathbb{S} = \{I = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mid x_1, x_2, x_3 \in \mathbb{R}, x_1^2 + x_2^2 + x_3^2 = 1\}$$

contains all the elements whose square is -1 . It is a 2-dimensional sphere in \mathbb{H} identified with \mathbb{R}^4 .

B. Quaternionic Hilbert spaces

In this subsection, we discuss right quaternionic Hilbert spaces. For more details, we refer the reader to Refs. 1, 13, and 23.

1. Right quaternionic Hilbert space

Let $V_{\mathbb{H}}^R$ be a vector space under right multiplication by quaternions. For $\phi, \psi, \omega \in V_{\mathbb{H}}^R$ and $q \in \mathbb{H}$, the inner product

$$\langle \cdot | \cdot \rangle_{V_{\mathbb{H}}^R} : V_{\mathbb{H}}^R \times V_{\mathbb{H}}^R \longrightarrow \mathbb{H}$$

satisfies the following properties:

- (i) $\overline{\langle \phi | \psi \rangle_{V_{\mathbb{H}}^R}} = \langle \psi | \phi \rangle_{V_{\mathbb{H}}^R}$,
- (ii) $\|\phi\|_{V_{\mathbb{H}}^R}^2 = \langle \phi | \phi \rangle_{V_{\mathbb{H}}^R} > 0$ unless $\phi = 0$, a real norm,
- (iii) $\langle \phi | \psi + \omega \rangle_{V_{\mathbb{H}}^R} = \langle \phi | \psi \rangle_{V_{\mathbb{H}}^R} + \langle \phi | \omega \rangle_{V_{\mathbb{H}}^R}$,
- (iv) $\langle \phi | \psi q \rangle_{V_{\mathbb{H}}^R} = \langle \phi | \psi \rangle_{V_{\mathbb{H}}^R} q$,
- (v) $\langle \phi q | \psi \rangle_{V_{\mathbb{H}}^R} = \bar{q} \langle \phi | \psi \rangle_{V_{\mathbb{H}}^R}$,

where \bar{q} stands for the quaternionic conjugate. It is always assumed that the space $V_{\mathbb{H}}^R$ is complete under the norm given above and separable. Then, together with $\langle \cdot | \cdot \rangle$, this defines a right quaternionic Hilbert space. Quaternionic Hilbert spaces share many of the standard properties of complex Hilbert spaces. In this paper, more than one Hilbert space will be involved in the results, but one notices that every Hilbert space, involved in results, is right quaternionic Hilbert space.

The next two propositions can be established following the proof of their complex counterparts; see, e.g., Refs. 13 and 23.

Proposition 2.1. Let $\mathcal{O} = \{\varphi_k \mid k \in N\}$ be an orthonormal subset of $V_{\mathbb{H}}^R$, where N is a countable index set. Then the following conditions are pairwise equivalent:

- (a) The closure of the linear combinations of elements in \mathcal{O} with coefficients on the right is $V_{\mathbb{H}}^R$.
- (b) For every $\phi, \psi \in V_{\mathbb{H}}^R$, the series $\sum_{k \in N} \langle \phi | \varphi_k \rangle_{V_{\mathbb{H}}^R} \langle \varphi_k | \psi \rangle_{V_{\mathbb{H}}^R}$ converges absolutely and it holds

$$\langle \phi | \psi \rangle_{V_{\mathbb{H}}^R} = \sum_{k \in N} \langle \phi | \varphi_k \rangle_{V_{\mathbb{H}}^R} \langle \varphi_k | \psi \rangle_{V_{\mathbb{H}}^R}.$$

- (c) For every $\phi \in V_{\mathbb{H}}^R$, it holds

$$\|\phi\|_{V_{\mathbb{H}}^R}^2 = \sum_{k \in N} |\langle \varphi_k | \phi \rangle_{V_{\mathbb{H}}^R}|^2.$$

- (d) $\mathcal{O}^\perp = \{0\}$.

Definition 2.2. The set \mathcal{O} as in Proposition 2.1 is called a Hilbert basis of $V_{\mathbb{H}}^R$.

Proposition 2.3. Every quaternionic Hilbert space $V_{\mathbb{H}}^R$ has a Hilbert basis. All the Hilbert bases of $V_{\mathbb{H}}^R$ have the same cardinality.

Furthermore, if \mathcal{O} is a Hilbert basis of $V_{\mathbb{H}}^R$, then every $\phi \in V_{\mathbb{H}}^R$ can be uniquely decomposed as follows:

$$\phi = \sum_{k \in N} \varphi_k \langle \varphi_k | \phi \rangle_{V_{\mathbb{H}}^R},$$

where the series $\sum_{k \in N} \varphi_k \langle \varphi_k | \phi \rangle_{V_{\mathbb{H}}^R}$ converges absolutely in $V_{\mathbb{H}}^R$.

It should be noted that once a Hilbert basis is fixed, every left (resp. right) quaternionic Hilbert space also becomes a right (resp. left) quaternionic Hilbert space.^{13,23}

The field of quaternions \mathbb{H} itself can be turned into a left quaternionic Hilbert space by defining the inner product $\langle q | q' \rangle = qq'$ or into a right quaternionic Hilbert space with $\langle q | q' \rangle = \bar{q}q'$.

III. RIGHT QUATERNIONIC LINEAR OPERATORS AND SOME BASIC PROPERTIES

In this section, we shall define right \mathbb{H} -linear operators and recall some basis properties. Most of them are very well known. In this manuscript, we follow the notations in Refs. 2 and 13. We shall also recall some results pertinent to the development of the paper.

Definition 3.1. A mapping $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$, where $\mathcal{D}(A)$ stands for the domain of A , is said to be right \mathbb{H} -linear operator or, for simplicity, right linear operator, if

$$A(\phi a + \psi b) = (A\phi)a + (A\psi)b, \quad \text{if } \phi, \psi \in \mathcal{D}(A) \text{ and } a, b \in \mathbb{H}.$$

The set of all right linear operators from $V_{\mathbb{H}}^R$ to $U_{\mathbb{H}}^R$ will be denoted by $\mathcal{L}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$, and the identity linear operator on $V_{\mathbb{H}}^R$ will be denoted by $\mathbb{I}_{V_{\mathbb{H}}^R}$. For a given $A \in \mathcal{L}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$, the range and the kernel will be

$$\begin{aligned} \text{ran}(A) &= \{\psi \in U_{\mathbb{H}}^R \mid A\phi = \psi \text{ for } \phi \in \mathcal{D}(A)\}, \\ \text{ker}(A) &= \{\phi \in \mathcal{D}(A) \mid A\phi = 0\}. \end{aligned}$$

We call an operator $A \in \mathcal{L}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ bounded if

$$\|A\| = \sup_{\|\phi\|_{V_{\mathbb{H}}^R}=1} \|A\phi\|_{U_{\mathbb{H}}^R} < \infty, \tag{3.1}$$

or equivalently, there exists $K \geq 0$ such that $\|A\phi\|_{U_{\mathbb{H}}^R} \leq K \|\phi\|_{V_{\mathbb{H}}^R}$ for all $\phi \in \mathcal{D}(A)$. The set of all bounded right linear operators from $V_{\mathbb{H}}^R$ to $U_{\mathbb{H}}^R$ will be denoted by $\mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$. The set of all invertible bounded right linear operators from $V_{\mathbb{H}}^R$ to $U_{\mathbb{H}}^R$ will be denoted by $\mathcal{G}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$. We also denote for a set $\Delta \subseteq \mathbb{H}$, $\Delta^* = \{\bar{q} \mid q \in \Delta\}$.

Assume that $V_{\mathbb{H}}^R$ is a right quaternionic Hilbert space and A is a right linear operator acting on it. Then, there exists a unique linear operator A^\dagger such that

$$\langle \psi | A\phi \rangle_{U_{\mathbb{H}}^R} = \langle A^\dagger \psi | \phi \rangle_{V_{\mathbb{H}}^R} \quad \text{for all } \phi \in \mathcal{D}(A), \psi \in \mathcal{D}(A^\dagger), \tag{3.2}$$

where the domain $\mathcal{D}(A^\dagger)$ of A^\dagger is defined by

$$\mathcal{D}(A^\dagger) = \{\psi \in U_{\mathbb{H}}^R \mid \exists \varphi \text{ such that } \langle \psi | A\phi \rangle_{U_{\mathbb{H}}^R} = \langle \varphi | \phi \rangle_{V_{\mathbb{H}}^R}\}.$$

The following theorem gives two important and fundamental results about right \mathbb{H} -linear bounded operators which are already appeared in Ref. 13 for the case of $V_{\mathbb{H}}^R = U_{\mathbb{H}}^R$. Point (b) of the following theorem is known as the open mapping theorem.

Theorem 3.2. Let $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$ be a right \mathbb{H} -linear operator. Then

- (a) $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ if and only if A is continuous;
- (b) if $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ is surjective, then A is open. In particular, if A is bijective, then $A^{-1} \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$.

Proof. The proof is the same as the proof in a complex Hilbert space (see, e.g., Ref. 8). □

The following proposition provides some useful aspects about the orthogonal complement subsets.

Proposition 3.3. Let $M \subseteq V_{\mathbb{H}}^R$. Then

- (a) M^\perp is closed;
- (b) if M is a closed subspace of $V_{\mathbb{H}}^R$, then $V_{\mathbb{H}}^R = M \oplus M^\perp$;
- (c) if $\dim(M) < \infty$, then M is a closed subspace.

Proof. The proof is similar to the proof of the complex case (see, e.g., Ref. 22). □

The points (a) and (b) of the following proposition are already appeared in Ref. 13 for the case $V_{\mathbb{H}}^R = U_{\mathbb{H}}^R$.

Proposition 3.4. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$. Then

- (a) $\text{ran}(A)^\perp = \ker(A^\dagger)$.
- (b) $\ker(A) = \text{ran}(A^\dagger)^\perp$.
- (c) $\ker(A)$ is a closed subspace of $V_{\mathbb{H}}^R$.

Proof. The proofs are elementary. □

Proposition 3.5. Let $A : \mathcal{D}(A) : V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$ be a right quaternionic linear operator. If A is closed and satisfies the condition that there exists $c > 0$ such that

$$\|A\phi\|_{U_{\mathbb{H}}^R} \geq c\|\phi\|_{V_{\mathbb{H}}^R}, \text{ for all } \phi \in \mathcal{D}(A),$$

then $\text{ran}(A)$ is closed.

Proof. The proof can be manipulated from the Proof of Proposition 2.13 in Ref. 19. □

Proposition 3.6. If $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ and if $\|A\| < 1$, then the right linear operator $\mathbb{I}_{V_{\mathbb{H}}^R} - A$ is invertible and the inverse is given by $(\mathbb{I}_{V_{\mathbb{H}}^R} - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

Proof. The proof is exactly the same as its complex version; see Theorem 2.17 in Ref. 10 for a complex proof. □

Theorem 3.7 (Bounded inverse theorem). Let $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$, and then the following results are equivalent:

- (a) A has a bounded inverse on its range.
- (b) A is bounded below.
- (c) A is injective and has a closed range.

Proof. The proof is exactly similar to its complex version. See Theorem 1.2 in Ref. 18 for a complex proof. □

Proposition 3.8. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$, and then $\text{ran}(A)$ is closed in $U_{\mathbb{H}}^R$ if and only if $\text{ran}(A^\dagger)$ is closed in $V_{\mathbb{H}}^R$.

Proof. The proof is the same as for the complex Hilbert spaces; see Lemma 1.5 in Ref. 18 for a complex proof. □

Proposition 3.9. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$. Then,

- (a) A is invertible if and only if it is injective with a closed range (i.e., $\ker(A) = \{0\}$ and $\overline{\text{ran}(A)} = \text{ran}(A)$);
- (b) A is left (right) invertible if and only if A^\dagger is right (left) invertible;
- (c) A is right invertible if and only if it is surjective (i.e., $\text{ran}(A) = V_{\mathbb{H}}^R$).

Proof. The proof is also exactly similar to its complex version. See Lemma 5.8 in Ref. 18 for a complex proof. □

Definition 3.10. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$. A closed subspace $M \subseteq V_{\mathbb{H}}^R$ is said to be invariant under A if $A(M) \subseteq M$, where $A(M) = \{A\phi \mid \phi \in M\}$.

The following results hold in any quaternionic normed linear space.

Definition 3.11. Let $X_{\mathbb{H}}^R$ be a quaternionic normed linear space. A subset C is said to be totally bounded, if for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and $\varphi_i \in C: i = 1, 2, 3, \dots, N$ such that

$$C \subseteq \bigcup_{i=1}^N B(\varphi_i; \varepsilon),$$

where $B(\varphi_i; \varepsilon) = \{\phi \in X_{\mathbb{H}}^R \mid \|\phi - \varphi_i\|_{X_{\mathbb{H}}^R} < \varepsilon\}$ for all $i = 1, 2, 3, \dots, N$.

Proposition 3.12. Let C be a subset in a right quaternionic normed linear space $X_{\mathbb{H}}^R$. Then

- (a) C is compact if and only if C is complete and totally bounded;
- (b) if $\dim(X_{\mathbb{H}}^R) < \infty$, C is compact if and only if C is closed and bounded.

Proof. The proof is exactly similar to the complex proof. For a complex proof of (a) and (b), see Theorem 6.19 in Ref. 21 and Theorems 2.5-3 in Ref. 17, respectively. □

A. Left scalar multiplications on $U_{\mathbb{H}}^R$

We shall extract the definition and some properties of left scalar multiples of vectors on $U_{\mathbb{H}}^R$ from Ref. 13, as needed for the development of the manuscript. The left scalar multiple of vectors in a right quaternionic Hilbert space is an extremely non-canonical operation associated with a choice of preferred Hilbert basis. From Proposition 2.3, $U_{\mathbb{H}}^R$ has a Hilbert basis

$$\mathcal{O} = \{\varphi_k \mid k \in N\}, \tag{3.3}$$

where N is a countable index set. The left scalar multiplication on $U_{\mathbb{H}}^R$ induced by \mathcal{O} is defined as the map $\mathbb{H} \times U_{\mathbb{H}}^R \ni (q, \phi) \mapsto q\phi \in U_{\mathbb{H}}^R$, given by

$$q\phi := \sum_{k \in N} \varphi_k q \langle \varphi_k \mid \phi \rangle, \tag{3.4}$$

for all $(q, \phi) \in \mathbb{H} \times U_{\mathbb{H}}^R$.

Proposition 3.13. (Ref. 13) The left product defined in Eq. (3.4) satisfies the following properties. For every $\phi, \psi \in U_{\mathbb{H}}^R$ and $p, q \in \mathbb{H}$,

- (a) $q(\phi + \psi) = q\phi + q\psi$ and $q(\phi p) = (q\phi)p$,
- (b) $\|q\phi\| = |q|\|\phi\|$,
- (c) $q(p\phi) = (qp)\phi$,
- (d) $\langle \bar{q}\phi | \psi \rangle = \langle \phi | q\psi \rangle$,
- (e) $r\phi = \phi r$, for all $r \in \mathbb{R}$,
- (f) $q\varphi_k = \varphi_k q$, for all $k \in N$.

Remark 3.14. (1) The meaning of writing $p\phi$ is $p \cdot \phi$ because the notation from Eq. (3.4) may be confusing, when $U_{\mathbb{H}}^R = \mathbb{H}$. However, regarding the field \mathbb{H} itself as a right \mathbb{H} -Hilbert space, an orthonormal basis \mathcal{O} should consist only of a singleton, say $\{\varphi_0\}$, with $|\varphi_0| = 1$ because we clearly have $\theta = \varphi_0\langle\varphi_0|\theta\rangle$, for all $\theta \in \mathbb{H}$. The equality from (f) of Proposition 3.13 can be written as $p\varphi_0 = \varphi_0p$, for all $p \in \mathbb{H}$. In fact, the left-hand side may be confusing and it should be understood as $p \cdot \varphi_0$ because the true equality $p\varphi_0 = \varphi_0p$ would imply that $\varphi_0 = \pm 1$. For the simplicity, we are writing $p\phi$ instead of writing $p \cdot \phi$.

(2) Also one can trivially see that $(p + q)\phi = p\phi + q\phi$, for all $p, q \in \mathbb{H}$ and $\phi \in U_{\mathbb{H}}^R$.

Furthermore, the quaternionic left scalar multiplication of linear operators is also defined in Refs. 5 and 13. For any fixed $q \in \mathbb{H}$ and a given right linear operator $A : V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$, the left scalar multiplication of A is defined as a map $qA : V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$ by the setting

$$(qA)\phi := q(A\phi) = \sum_{k \in N} \varphi_k q \langle \varphi_k | A\phi \rangle, \tag{3.5}$$

for all $\phi \in V_{\mathbb{H}}^R$. It is straightforward that qA is a right linear operator. If $q\phi \in V_{\mathbb{H}}^R$, for all $\phi \in V_{\mathbb{H}}^R$, one can define the right scalar multiplication of the right linear operator $A : V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$ as a map $Aq : V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$ by the setting

$$(Aq)\phi := A(q\phi), \tag{3.6}$$

for all $\phi \in D(A)$. It is also a right linear operator. One can easily obtain that, if $q\phi \in V_{\mathbb{H}}^R$, for all $\phi \in V_{\mathbb{H}}^R$, and $V_{\mathbb{H}}^R$ is dense in $U_{\mathbb{H}}^R$, then

$$(qA)^\dagger = A^\dagger \bar{q} \text{ and } (Aq)^\dagger = \bar{q}A^\dagger. \tag{3.7}$$

B. S-spectrum

For a given right linear operator $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \rightarrow V_{\mathbb{H}}^R$ and $q \in \mathbb{H}$, we define the operator $R_q(A) : \mathcal{D}(A^2) \rightarrow \mathbb{H}$ by

$$R_q(A) = A^2 - 2\text{Re}(q)A + |q|^2 \mathbb{I}_{V_{\mathbb{H}}^R},$$

where $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ is a quaternion, $\text{Re}(q) = q_0$, and $|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$. In the literature, the operator is called pseudo-resolvent since it is not the resolvent operator of A but it is the one related to the notion of spectrum as we shall see in the next definition. For more information, on the notion of S -spectrum, the reader may consult, e.g., Refs. 4, 5, 7, and 13.

Definition 3.15. Let $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \rightarrow V_{\mathbb{H}}^R$ be a right linear operator. The S -resolvent set (also called spherical resolvent set) of A is the set $\rho_S(A) (\subset \mathbb{H})$ such that the three following conditions hold true:

- (a) $\ker(R_q(A)) = \{0\}$.
- (b) $\text{ran}(R_q(A))$ is dense in $V_{\mathbb{H}}^R$.
- (c) $R_q(A)^{-1} : \text{ran}(R_q(A)) \rightarrow \mathcal{D}(A^2)$ is bounded.

The S -spectrum (also called spherical spectrum) $\sigma_S(A)$ of A is defined by setting $\sigma_S(A) := \mathbb{H} \setminus \rho_S(A)$. For a bounded linear operator A , we can write the resolvent set as

$$\begin{aligned} \rho_S(A) &= \{q \in \mathbb{H} \mid R_q(A) \in \mathcal{G}(V_{\mathbb{H}}^R)\} \\ &= \{q \in \mathbb{H} \mid R_q(A) \text{ has an inverse in } \mathcal{B}(V_{\mathbb{H}}^R)\} \\ &= \{q \in \mathbb{H} \mid \ker(R_q(A)) = \{0\} \text{ and } \text{ran}(R_q(A)) = V_{\mathbb{H}}^R\}, \end{aligned}$$

and the spectrum can be written as

$$\begin{aligned} \sigma_S(A) &= \mathbb{H} \setminus \rho_S(A) \\ &= \{q \in \mathbb{H} \mid R_q(A) \text{ has no inverse in } \mathcal{B}(V_{\mathbb{H}}^R)\} \\ &= \{q \in \mathbb{H} \mid \ker(R_q(A)) \neq \{0\} \text{ or } \text{ran}(R_q(A)) \neq V_{\mathbb{H}}^R\}. \end{aligned}$$

The right S -spectrum $\sigma_r^S(A)$ and the left S -spectrum $\sigma_l^S(A)$ are defined, respectively, as

$$\begin{aligned} \sigma_r^S(A) &= \{q \in \mathbb{H} \mid R_q(A) \text{ in not right invertible in } \mathcal{B}(V_{\mathbb{H}}^R)\}, \\ \sigma_l^S(A) &= \{q \in \mathbb{H} \mid R_q(A) \text{ in not left invertible in } \mathcal{B}(V_{\mathbb{H}}^R)\}. \end{aligned}$$

The spectrum $\sigma_S(A)$ decomposes into three major disjoint subsets as follows:

- (i) The *spherical point spectrum* of A :

$$\sigma_{pS}(A) := \{q \in \mathbb{H} \mid \ker(R_q(A)) \neq \{0\}\}.$$

- (ii) The *spherical residual spectrum* of A :

$$\sigma_{rS}(A) := \{q \in \mathbb{H} \mid \ker(R_q(A)) = \{0\}, \overline{\text{ran}(R_q(A))} \neq V_{\mathbb{H}}^R\}.$$

- (iii) The *spherical continuous spectrum* of A :

$$\sigma_{cS}(A) := \{q \in \mathbb{H} \mid \ker(R_q(A)) = \{0\}, \overline{\text{ran}(R_q(A))} = V_{\mathbb{H}}^R, R_q(A)^{-1} \notin \mathcal{B}(V_{\mathbb{H}}^R)\}.$$

If $A\phi = \phi q$ for some $q \in \mathbb{H}$ and $\phi \in V_{\mathbb{H}}^R \setminus \{0\}$, then ϕ is called an eigenvector of A with the right eigenvalue q . The set of right eigenvalues coincides with the point S -spectrum; see Ref. 13, Proposition 4.5.

Proposition 3.16 (Refs. 6 and 13). For $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, the resolvent set $\rho_S(A)$ is a non-empty open set and the spectrum $\sigma_S(A)$ is a non-empty compact set.

Remark 3.17. For $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, since $\sigma_S(A)$ is a non-empty compact set so is its boundary. That is, $\partial\sigma_S(A) = \partial\rho_S(A) \neq \emptyset$.

Proposition 3.18. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$,

$$\sigma_l^S(A) = \{q \in \mathbb{H} \mid \text{ran}(R_q(A)) \text{ is closed or } \ker(R_q(A)) \neq \{0\}\}, \tag{3.8}$$

$$\sigma_r^S(A) = \{q \in \mathbb{H} \mid \text{ran}(R_q(A)) \text{ is closed or } \ker(R_q(A^\dagger)) \neq \{0\}\}. \tag{3.9}$$

Proof. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $q \in \mathbb{H}$. Set $S = R_q(A) \in \mathcal{B}(V_{\mathbb{H}}^R)$. By Proposition 3.9, S is not left invertible if and only if $\text{ran}(S)$ is not closed or $\ker(S) \neq \{0\}$. Thus we have Eq. (3.8). Again by Proposition 3.9, A is not right invertible if and only if $\text{ran}(S) \neq V_{\mathbb{H}}^R$. $\text{ran}(S) \neq V_{\mathbb{H}}^R$ if and only if $\overline{\text{ran}(S)} \neq \text{ran}(S)$ or $\overline{\text{ran}(S)} \neq V_{\mathbb{H}}^R$. That is, $\text{ran}(S) \neq V_{\mathbb{H}}^R$ if and only if $\text{ran}(S) \neq \overline{\text{ran}(S)}$ or $\ker(S^\dagger)^\perp \neq V_{\mathbb{H}}^R$. Hence we have Eq. (3.9). □

IV. QUATERNIONIC COMPACT OPERATORS

A systematic study of quaternionic compact operators has not appeared in the literature. In this regard, in this section, as needed for our purpose, we provide certain significant results about compact right linear operators on $V_{\mathbb{H}}^R$.

Definition 4.1. Let $V_{\mathbb{H}}^R$ and $U_{\mathbb{H}}^R$ be right quaternionic Hilbert spaces. A bounded operator $K : V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$ is compact if K maps bounded sets into precompact sets. That is, $\overline{K(U)}$ is compact in $U_{\mathbb{H}}^R$, where $U = \{\phi \in V_{\mathbb{H}}^R \mid \|\phi\|_{V_{\mathbb{H}}^R} \leq 1\}$. Equivalently, for all bounded sequences $\{\phi_n\}_{n=1}^{\infty}$ in $V_{\mathbb{H}}^R$, the sequence $\{K\phi_n\}_{n=0}^{\infty}$ has a convergence subsequence in $U_{\mathbb{H}}^R$.¹¹

We denote the set of all compact operators from $V_{\mathbb{H}}^R$ to $U_{\mathbb{H}}^R$ by $\mathcal{B}_0(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ and the compact operators from $V_{\mathbb{H}}^R$ will be denoted by $\mathcal{B}_0(V_{\mathbb{H}}^R)$.

Proposition 4.2. The following statements are true:

- (a) If $(q, A) \in \mathbb{H} \times \mathcal{B}_0(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$, then $qA, Aq \in \mathcal{B}_0(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ which are defined by Eqs. (3.5) and (3.6), respectively.
- (b) $\mathcal{B}_0(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ is a vector space under left-scalar multiplication.
- (c) If $\{A_n\} \subseteq \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ and $A \in \mathcal{B}_0(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ such that $\|A_n - A\| \rightarrow 0$, then $A \in \mathcal{B}_0(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$.
- (d) If $A \in \mathcal{B}(U_{\mathbb{H}}^R)$, $B \in \mathcal{B}(U_{\mathbb{H}}^R)$, and $K \in \mathcal{B}_0(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$, then AK and KB are compact operators.

Proof. Let $\{\phi_n\} \subseteq V_{\mathbb{H}}^R$ be a bounded sequence. Then $\{A\phi_n\}$ has a convergent subsequence. Thus $\{qA\phi_n\}$ also has a convergent subsequence, and so qA is a compact operator. Now $\{q\phi_n\} \subseteq V_{\mathbb{H}}^R$ is also a bounded sequence. Thus $\{(Aq)\phi_n\}$ has a convergent subsequence, as A is compact. Therefore point (a) follows. And point (b) is straightforward.

Now suppose that $\{A_n\} \subseteq \mathcal{B}_0(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ and $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ such that $\|A_n - A\| \rightarrow 0$ and let $\varepsilon > 0$. Then by Proposition 3.12, it is enough to show that $\overline{A(U)}$ is totally bounded, as $\overline{A(U)}$ is a complete set, where $U = \{\phi \in V_{\mathbb{H}}^R \mid \|\phi\|_{V_{\mathbb{H}}^R} \leq 1\}$. Let $\varepsilon > 0$, and then there exists $N \in \mathbb{N}$ such that $\|A - A_n\| < \varepsilon/6$, for all $n > N$. Since A_n is compact, there is $\phi_j \in U: j = 1, 2, 3, \dots, m$ such that

$$A_n(U) \subseteq \overline{A_n(U)} \subseteq \bigcup_{i=1}^m B(A_n\phi_j; \varepsilon/6),$$

where $B(A_n\phi_j; \varepsilon/6) = \{A_n\phi \mid \|A_n\phi - A_n\phi_j\|_{U_{\mathbb{H}}^R} < \varepsilon/6\}$, for all $j = 1, 2, 3, \dots, m$ and $n \in \mathbb{N}$. Let $\psi \in \overline{A(U)}$. Then there exists a sequence $\{\phi^{(\ell)}\} \subseteq U$ such that $A\phi^{(\ell)} \rightarrow \psi$ as $\ell \rightarrow \infty$. So for each $(n, \ell) \in \mathbb{N}^2$, there is $\phi_j \in U$ such that $\|A_n\phi_j - A_n\phi^{(\ell)}\|_{U_{\mathbb{H}}^R} < \varepsilon/6$ as $A_n\phi^{(\ell)} \in A_n(U)$, for all $(n, \ell) \in \mathbb{N}^2$. Therefore, for each $(n, \ell) \in \mathbb{N}^2$ with $n > N$,

$$\begin{aligned} \|A\phi_j - A\phi^{(\ell)}\|_{U_{\mathbb{H}}^R} &\leq \|A\phi_j - A_n\phi_j\|_{U_{\mathbb{H}}^R} + \|A_n\phi_j - A_n\phi^{(\ell)}\|_{U_{\mathbb{H}}^R} + \|A_n\phi^{(\ell)} - A\phi^{(\ell)}\|_{U_{\mathbb{H}}^R} \\ &< 2\|A - A_n\| + \varepsilon/6 \\ &< \varepsilon/2. \end{aligned}$$

Now taking limit $\ell \rightarrow \infty$ on both sides of the inequality $\|A\phi_j - A\phi^{(\ell)}\|_{U_{\mathbb{H}}^R} < \varepsilon/2$ gives $\|A\phi_j - \psi\|_{U_{\mathbb{H}}^R} \leq \varepsilon/2 < \varepsilon$ as the norm $\|\cdot\|$ is a continuous map. Thus

$$\overline{A(U)} \subseteq \bigcup_{i=1}^m B(A\phi_j; \varepsilon).$$

This concludes the result of (c).

Now suppose that $A \in \mathcal{B}(U_{\mathbb{H}}^R)$, $B \in \mathcal{B}(V_{\mathbb{H}}^R)$, and $K \in \mathcal{B}_0(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$. Then for any bounded sequence $\{\phi_n\} \subseteq V_{\mathbb{H}}^R$, $\{K\phi_n\} \subseteq U_{\mathbb{H}}^R$ has a convergent subsequence $\{K\phi_{n_k}\}$, as K is compact. Thus $\{AK\phi_{n_k}\}$ is a convergent subsequence of $\{AK\phi_n\}$ because A is continuous. That is, AK is compact. In a similar fashion, we can obtain that KB is compact. Hence the results hold. \square

Definition 4.3. An operator $A : V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$ is said to be of finite rank if $\text{ran}(A) \subseteq U_{\mathbb{H}}^R$ is finite dimensional.

Lemma 4.4. The following statements are equivalent:

- (a) *The operator $A : V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$ is a finite rank operator.*
- (b) *For each $\phi \in V_{\mathbb{H}}^R$, there exists $(v_i, u_i) \in V_{\mathbb{H}}^R \times U_{\mathbb{H}}^R : i = 1, 2, 3, \dots, n$ such that*

$$\langle u_i | u_j \rangle_{U_{\mathbb{H}}^R} = \delta_{ij} \text{ and } A\phi = \sum_{i=1}^n u_i \langle v_i | \phi \rangle_{V_{\mathbb{H}}^R}, \tag{4.1}$$

where δ_{ij} is the Kronecker delta.

Proof. Suppose that (a) holds. Then $\dim(\text{ran}(A)) < \infty$, and so there exists an orthonormal basis $e_i : i = 1, 2, 3, \dots, n$ for $\text{ran}(A)$. Thus for each $\phi \in V_{\mathbb{H}}^R$,

$$A\phi = \sum_{i=1}^n e_i \langle e_i | A\phi \rangle_{U_{\mathbb{H}}^R} = \sum_{i=1}^n e_i \langle A^\dagger e_i | \phi \rangle_{V_{\mathbb{H}}^R}.$$

Point (b) follows as taking $u_i = e_i$ and $v_i = A^\dagger e_i$. On the other hand, suppose (b) holds. Then it immediately follows that $\text{ran}(A) = \text{right span over } \mathbb{H}\{u_i \mid i = 1, 2, 3, \dots, n\}$ as $u_i \in \text{ran}(A)$, for all $i = 1, 2, 3, \dots, n$. Assume that

$$\exists q_i \in \mathbb{H} : i = 1, 2, 3, \dots, n \text{ such that } \sum_{i=1}^n u_i q_i = 0.$$

Since $\sum_{i=1}^n (A^\dagger u_i) q_i = 0$ and there exists $\phi_j \in V_{\mathbb{H}}^R$ such that $u_i = A\phi_j$, for all $i = 1, 2, 3, \dots, n$, we have for each $j = 1, 2, 3, \dots, n$,

$$\bar{q}_j = \sum_{i=1}^n \bar{q}_i \langle u_i | A\phi_j \rangle_{U_{\mathbb{H}}^R} = \sum_{i=1}^n \langle (A^\dagger u_i) q_i | \phi_j \rangle_{V_{\mathbb{H}}^R} = 0.$$

That is, the set $\{u_i \mid i = 1, 2, 3, \dots, n\}$ is linearly independent. Thus $\{u_i \mid i = 1, 2, 3, \dots, n\}$ is an orthonormal basis for $\text{ran}(A)$, and the point (a) holds. Hence the lemma follows. \square

Lemma 4.5. In Lemma 4.4, the set $\{u_i \mid i = 1, 2, 3, \dots, n\}$ of vectors can be chosen to be an orthonormal basis for $\text{ran}(A)$, and the adjoint of A can be written as

$$A^\dagger \psi = \sum_{i=1}^n v_i \langle u_i | \psi \rangle_{U_{\mathbb{H}}^R}, \tag{4.2}$$

for all $\psi \in U_{\mathbb{H}}^R$. Furthermore, $A^\dagger u_i = v_i$ for all $i = 1, 2, 3, \dots, n$ and the set $\{v_i \mid i = 1, 2, 3, \dots, n\}$ of vectors can be chosen to be a basis for $\text{ran}(A^\dagger)$.

Proof. In Lemma 4.4, one can trivially observe that the set $\{u_i \mid i = 1, 2, 3, \dots, n\}$ of vectors can be chosen to be an orthonormal basis for $\text{ran}(A)$. Now let $(\phi, \psi) \in V_{\mathbb{H}}^R \times U_{\mathbb{H}}^R$, and from Eq. (4.1), we have

$$\langle \psi | A\phi \rangle_{U_{\mathbb{H}}^R} = \sum_{i=1}^n \langle \psi | u_i \rangle_{U_{\mathbb{H}}^R} \langle v_i | \phi \rangle_{V_{\mathbb{H}}^R}.$$

This implies that $\langle A^\dagger \psi | \phi \rangle_{V_{\mathbb{H}}^R} = \langle \sum_{i=1}^n v_i \langle \psi | u_i \rangle_{U_{\mathbb{H}}^R} | \phi \rangle_{V_{\mathbb{H}}^R}$, and so Eq. (4.2) holds. From Eq. (4.2), it immediately follows that $A^\dagger u_i = v_i$ for all $i = 1, 2, 3, \dots, n$, as $\langle u_i | u_j \rangle_{U_{\mathbb{H}}^R} = \delta_{ij}$. This guarantees that for each $i = 1, 2, 3, \dots, n$, $v_i \in \text{ran}(A^\dagger)$, and it suffices, together with (4.2), to say that $\text{ran}(A^\dagger) = \text{right span over } \mathbb{H}\{v_i \mid i = 1, 2, 3, \dots, n\}$. Now assume that

$$\exists p_i \in \mathbb{H} : i = 1, 2, 3, \dots, n \text{ such that } \sum_{i=1}^n v_i p_i = 0.$$

Then $\sum_{i=1}^n (A^\dagger u_i) p_i = 0$ and $\sum_{i=1}^n u_i p_i \in \ker(A^\dagger) = \text{ran}(A)^\perp$ by (a) of Proposition 3.4. Thus $\sum_{i=1}^n u_i p_i = 0$ as $u_i \in \text{ran}(A)$, for all $i = 1, 2, 3, \dots, n$. Since $\{u_i \mid i = 1, 2, 3, \dots, n\}$ forms a (orthonormal) basis,

we have for each $i = 1, 2, 3, \dots, n, p_i = 0$. That is, $\{v_i \mid i = 1, 2, 3, \dots, n\}$ is a linear independent set in $\text{ran}(A^\dagger)$. This completes the proof. \square

Theorem 4.6. *Let $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ be a finite rank operator. Then*

- (a) *A is compact;*
- (b) *$A^\dagger \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ is a finite rank operator and $\dim(\text{ran}(A)) = \dim(\text{ran}(A^\dagger))$.*

Proof. Since the operator $A : V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$ is a finite rank, we have $\dim(A(U)) < \dim(\text{ran}(A)) < \infty$, where $U = \{\phi \in V_{\mathbb{H}}^R \mid \|\phi\|_{V_{\mathbb{H}}^R} \leq 1\}$. Hence by Proposition 3.3, $A(U)$ is closed. Now for each $\phi \in U$,

$$\|A\phi\|_{U_{\mathbb{H}}^R} \leq \|A\| \|\phi\|_{V_{\mathbb{H}}^R} \leq \|A\|.$$

That is, $A(U)$ is a closed and bounded set. Hence by Proposition 3.12, $\overline{A(U)} = A(U)$ is compact, and so A is a compact operator. This concludes the point (a). From Lemmas 4.4 and 4.5, the point (b) immediately follows. \square

Definition 4.7. Let $M \subset V_{\mathbb{H}}^R$ be a closed subspace, and then $\text{codim}(M) = \dim(V_{\mathbb{H}}^R/M)$.

Definition 4.8. Let $A : V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$ be a bounded operator, and then $\text{coker}(A) := U_{\mathbb{H}}^R/\text{ran}(A)$ and $\dim(\text{coker}(A)) = \dim(U_{\mathbb{H}}^R) - \dim(\text{ran}(A))$.

Theorem 4.9. *If $A \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$, then the following statements are equivalent:*

- (a) *A is compact.*
- (b) *There exist finite rank operators $A_n : V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$ such that $\|A - A_n\| \rightarrow 0$ as $n \rightarrow \infty$.*
- (c) *A^\dagger is compact.*

Proof. Suppose that point (a) holds. Then $\overline{A(U)}$ is a compact set, where $U = \{\phi \in V_{\mathbb{H}}^R \mid \|\phi\|_{V_{\mathbb{H}}^R} \leq 1\}$. So $\overline{A(U)}$ is separable, and there exists an orthonormal basis $\{e_i \mid i = 1, 2, 3, \dots\}$ for $\overline{A(U)}$. Now, for any $n \in \mathbb{N}$, define $A_n : V_{\mathbb{H}}^R \rightarrow U_{\mathbb{H}}^R$ by

$$A_n\phi = \sum_{i=1}^n e_i \langle A^\dagger e_i \mid \phi \rangle_{V_{\mathbb{H}}^R}, \text{ for all } \phi \in V_{\mathbb{H}}^R.$$

Then by Lemma 4.4, A_n is a finite rank operator, for all $n \in \mathbb{N}$. Furthermore, for every $\phi \in U$, we have

$$\|A\phi - A_n\phi\|_{U_{\mathbb{H}}^R}^2 = \left\| \sum_{i>n} e_i \langle A^\dagger e_i \mid \phi \rangle_{V_{\mathbb{H}}^R} \right\|_{U_{\mathbb{H}}^R}^2 = \left\| \sum_{i>n} e_i \langle e_i \mid A\phi \rangle_{U_{\mathbb{H}}^R} \right\|_{U_{\mathbb{H}}^R}^2 = \sum_{i>n} |\langle e_i \mid A\phi \rangle_{U_{\mathbb{H}}^R}|^2.$$

Thus $\|A\phi - A_n\phi\|_{U_{\mathbb{H}}^R} \rightarrow 0$ as $n \rightarrow \infty$, for all $\phi \in U$, since for each $\phi \in U$,

$$\sum_{i>n} |\langle e_i \mid A\phi \rangle_{U_{\mathbb{H}}^R}|^2 = \left| \sum_{i=1}^{\infty} |\langle e_i \mid A\phi \rangle_{U_{\mathbb{H}}^R}|^2 - \sum_{i=1}^n |\langle e_i \mid A\phi \rangle_{U_{\mathbb{H}}^R}|^2 \right| \rightarrow 0$$

as $n \rightarrow \infty$. Therefore,

$$\|A - A_n\| = \sup_{\|\phi\|_{V_{\mathbb{H}}^R}=1} \|A\phi - A_n\phi\|_{U_{\mathbb{H}}^R} \rightarrow 0$$

as $n \rightarrow \infty$. Hence point (b) holds.

Suppose that point (b) holds. Then $\|A^\dagger - A_n^\dagger\| = \|A - A_n\| \rightarrow 0$ as $n \rightarrow \infty$. Thus point (c) follows from Theorem 4.6 and (c) of Proposition 4.2.

The implication (c) \Rightarrow (a) can be obtained, by applying the implications (a) \Rightarrow (b) and (b) \Rightarrow (c) for A^\dagger and using the fact that $A^{\dagger\dagger} = A$. Hence the theorem follows. \square

Proposition 4.10. If $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ is compact, then $\dim(\ker(q\mathbb{I}_{V_{\mathbb{H}}^R} - A)) < \infty$, for all $q \in \mathbb{H} \setminus \{0\}$.

Proof. Let $q \in \mathbb{H} \setminus \{0\}$. Assume that $\dim(\ker(q\mathbb{I}_{V_{\mathbb{H}}^R} - A)) = \infty$. Then there exists a sequence $\{\phi_n\} \subseteq \ker(q\mathbb{I}_{V_{\mathbb{H}}^R} - A)$ such that

$$\|\phi_n\|_{V_{\mathbb{H}}^R} = 1 \text{ and } \|\phi_n - \phi_m\|_{V_{\mathbb{H}}^R} > \frac{1}{2}, \text{ for any } m, n \in \mathbb{N} \text{ with } m \neq n,$$

as $\ker(q\mathbb{I}_{V_{\mathbb{H}}^R} - A)$ is separable. Now for each $n \in \mathbb{N}$, $A\phi_n = q\phi_n$ implies

$$\|A\phi_n\|_{V_{\mathbb{H}}^R} = |q| \text{ and } \|A\phi_n - A\phi_m\|_{V_{\mathbb{H}}^R} > \frac{|q|}{2}, \text{ for any } m, n \in \mathbb{N} \text{ with } m \neq n.$$

This suffices to say that A is not compact. This concludes the result. □

V. APPROXIMATE S-POINT SPECTRUM

Following the complex case, we define the approximate spherical point spectrum of $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ as follows:

Definition 5.1. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$. The approximate S-point spectrum of A , denoted by $\sigma_{ap}^S(A)$, is defined as

$$\sigma_{ap}^S(A) = \{q \in \mathbb{H} \mid \text{there is a sequence } \{\phi_n\}_{n=1}^{\infty} \text{ such that } \|\phi_n\| = 1 \text{ and } \|R_q(A)\phi_n\| \rightarrow 0\}.$$

Proposition 5.2. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, and then $\sigma_{pS}(A) \subseteq \sigma_{ap}^S(A)$.

Proof. Let $q \in \sigma_{pS}(A)$, and then $\ker(R_q(A)) \neq \{0\}$. That is, $\{\phi \in \mathcal{D}(A^2) \mid R_q(A)\phi = 0\} \neq \{0\}$. Therefore, there exist $\phi \in \mathcal{D}(A^2)$ and $\phi \neq 0$ such that $R_q(A)\phi = 0$. Set $\phi_n = \phi/\|\phi\|$ for all $n = 1, 2, \dots$. Then $\|\phi_n\| = 1$ for all n and $\|R_q(A)\phi_n\| \rightarrow 0$ as $n \rightarrow \infty$. Thus $q \in \sigma_{ap}^S(A)$. □

Proposition 5.3. If $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $q \in \mathbb{H}$, then the following statements are equivalent:

- (a) $q \notin \sigma_{ap}^S(A)$.
- (b) $\ker(R_q(A)) = \{0\}$ and $\text{ran}(R_q(A))$ is closed.
- (c) There exists a constant $c \in \mathbb{R}$, $c > 0$ such that $\|R_q(A)\phi\| \geq c\|\phi\|$ for all $\phi \in \mathcal{D}(A^2)$.

Proof. (a) \Rightarrow (c): Suppose (c) fails to hold. Then for every n , there exist non-zero vectors ϕ_n such that

$$\|R_q(A)\phi_n\| \leq \frac{\|\phi_n\|}{n}.$$

For each n , let $\psi_n = \frac{\phi_n}{\|\phi_n\|}$, and then $\|\psi_n\| = 1$ and $\|R_q(A)\psi_n\| < \frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$. Therefore $q \in \sigma_{ap}^S(A)$.

(c) \Rightarrow (b): Suppose that there is a constant $c > 0$ such that $\|R_q(A)\phi\| \geq c\|\phi\|$ for all $\phi \in \mathcal{D}(A^2)$. Therefore, $R_q(A)\phi \neq 0$ for all $\phi \neq 0$ and $R_q(A)0 = 0$. Therefore $\ker(R_q(A)) = \{0\}$. Let $\phi \in \overline{\text{ran}(R_q(A))}$, where the bar stands for the closure. Then there exists a sequence $\{\phi_n\} \subseteq \text{ran}(R_q(A))$ such that $\phi_n \rightarrow \phi$ as $n \rightarrow \infty$. From (c), we have a constant $c > 0$ such that $\|\phi_n - \phi_m\| \leq \frac{1}{c}\|R_q(A)\phi_n - R_q(A)\phi_m\|$. Therefore $\{\phi_n\}$ is a Cauchy sequence as $\{R_q(A)\phi_n\}$ is a Cauchy sequence. Let $\psi = \lim_{n \rightarrow \infty} \phi_n$, and then $R_q(A)\psi = \lim_{n \rightarrow \infty} R_q(A)\phi_n$ as R_q is continuous. Therefore $R_q(A)\psi = \phi$, and hence $\phi \in \text{ran}(R_q(A))$, from which we get $\overline{\text{ran}(R_q(A))} \subseteq \text{ran}(R_q(A))$. Hence $\text{ran}(R_q(A))$ is closed.

(b) \Rightarrow (a): Suppose that $\ker(R_q(A)) = \{0\}$ and $\text{ran}(R_q(A))$ is closed. Since $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, $R_q(A)$ is bounded and $R_q(A) : \mathcal{D}(A^2) \rightarrow \text{ran}(R_q(A))$ is a bijection. Therefore, by Theorem 3.2, there is a bounded linear operator $B : \text{ran}(R_q(A)) \rightarrow \mathcal{D}(A^2)$ such that $BR_q(A)\phi = \phi$ for all $\phi \in \mathcal{D}(A^2)$. Thus,

$\|\phi\| = \|BR_q(A)\phi\| \leq \|B\|\|R_q(A)\phi\|$ for all $\phi \in \mathcal{D}(A^2)$. Therefore, if there is a sequence $\{\phi_n\} \subseteq \mathcal{D}(A^2)$ with $\|\phi_n\| = 1$, then $1 \leq \|B\|\|R_q(A)\phi_n\|$. That is, $\frac{1}{\|B\|} \leq \|R_q(A)\phi_n\|$. Hence, there is no sequence $\{\phi_n\} \subseteq \mathcal{D}(A^2)$ with $\|\phi_n\| = 1$ such that $\|R_q(A)\phi_n\| \rightarrow 0$ as $n \rightarrow \infty$. Therefore $q \notin \sigma_{ap}^S(A)$. \square

Theorem 5.4. *Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, and then $\sigma_{ap}^S(A)$ is a non-empty closed subset of \mathbb{H} and $\partial\sigma_S(A) \subseteq \sigma_{ap}^S(A)$, where $\partial\sigma_S(A)$ is the boundary of $\sigma_S(A)$.*

Proof. Let $q \in \partial\sigma_S(A)$. Since $\rho_S(A) \neq \emptyset$ and $\sigma_S(A)$ is closed,

$$\partial\sigma_S(A) = \partial\rho_S(A) = \overline{\rho_S(A)} \cap \sigma_S(A).$$

Therefore, there exists a sequence $\{q_n\} \subseteq \rho_S(A)$ such that $q_n \rightarrow q$ as $n \rightarrow \infty$. Since,

$$R_{q_n}(A) - R_q(A) = (\|q_n\| - \|q\|)\mathbb{I}_{V_{\mathbb{H}}^R} - 2(\operatorname{Re}(q_n) - \operatorname{Re}(q))A, \quad \text{for all } n,$$

we have $R_{q_n}(A) \rightarrow R_q(A)$ in $\mathcal{B}(V_{\mathbb{H}}^R)$ as $\|q_n\| \rightarrow \|q\|$ and $\operatorname{Re}(q_n) \rightarrow \operatorname{Re}(q)$.

Claim. $\sup_n \|R_{q_n}(A)^{-1}\| = \infty$. Since $\sigma_S(A)$ is closed, $q_n \in \rho_S(A)$, $q \in \partial\sigma_S(A)$, and $q \notin \rho_S(A)$, we have $R_{q_n}(A) \rightarrow R_q(A)$ in $\mathcal{B}(V_{\mathbb{H}}^R)$ and $R_q(A) \in \mathcal{B}(V_{\mathbb{H}}^R) \setminus \mathcal{G}(V_{\mathbb{H}}^R)$.

Case-I: $q \in \sigma_{ap}^S(A)$. Assume $\sup_n \|R_{q_n}(A)^{-1}\| < \infty$. Since $q \in \sigma_{ap}^S(A)$, $\ker(R_q(A)) \neq \{0\}$. Therefore, there exists $\phi \in \mathcal{D}(A^2)$ with $\phi \neq 0$ such that $R_q(A)\phi = 0$. Thus,

$$\begin{aligned} 0 \neq \|\phi\| &= \|R_{q_n}(A)^{-1}R_{q_n}(A)\phi\| \\ &\leq \sup_n \|R_{q_n}(A)^{-1}\| \limsup_n \|R_{q_n}(A)\phi\| \\ &\leq \sup_n \|R_{q_n}(A)^{-1}\| \|R_q(A)\phi\| = 0, \end{aligned}$$

which is a contradiction. Therefore $\sup_n \|R_{q_n}(A)^{-1}\| = \infty$.

Case-II: $q \notin \sigma_{ap}^S(A)$. Then $\ker(R_q(A)) = \{0\}$. Thus $R_q(A)^{-1} : \operatorname{ran}(R_q(A)) \rightarrow V_{\mathbb{H}}^R$ exists. For each n , we have

$$R_{q_n}(A)^{-1} - R_q(A)^{-1} = R_{q_n}(A)^{-1}[R_q(A) - R_{q_n}(A)]R_q(A)^{-1}.$$

Assume $\sup_n \|R_{q_n}(A)^{-1}\| < \infty$. Then, since $R_{q_n}(A) \rightarrow R_q(A)$ in $\mathcal{B}(V_{\mathbb{H}}^R)$, $R_{q_n}(A)^{-1} \rightarrow R_q(A)^{-1}$ and $R_q(A)^{-1} \in \mathcal{B}(V_{\mathbb{H}}^R)$, that is, $R_q(A) \in \mathcal{G}(V_{\mathbb{H}}^R)$, which is a contradiction to $q \in \sigma_S(A)$. Therefore, $\sup_n \|R_{q_n}(A)^{-1}\| = \infty$. The claim is proved. Since $\|R_{q_n}(A)^{-1}\| = \sup_{\|\psi\|=1} \|R_{q_n}(A)^{-1}\psi\|$, we can take vectors $\phi_n \in V_{\mathbb{H}}^R$ with $\|\phi_n\| = 1$ such that, for each n ,

$$\|R_{q_n}(A)^{-1}\| - \frac{1}{n} \leq \|R_{q_n}(A)^{-1}\phi_n\| \leq \|R_{q_n}(A)^{-1}\|.$$

From which, we have $\sup_n \|R_{q_n}(A)^{-1}\phi_n\| = \infty$, and hence, $\inf_n \|R_{q_n}(A)^{-1}\phi_n\|^{-1} = 0$. Therefore, there are subsequences of $\{\phi_n\}$ and $\{q_n\}$, call them $\{\phi_k\}$ and $\{q_k\}$ such that

$$\|R_{q_k}(A)^{-1}\phi_k\|^{-1} \rightarrow 0, \quad \text{as } k \rightarrow \infty. \tag{5.1}$$

Set $\psi_k = \frac{R_{q_k}(A)^{-1}\phi_k}{\|R_{q_k}(A)^{-1}\phi_k\|}$, and then $\{\psi_k\}$ is a sequence of unit vectors in $V_{\mathbb{H}}^R$ with

$$\|R_{q_k}(A)\psi_k\| = \frac{\|R_{q_k}(A)R_{q_k}(A)^{-1}\phi_k\|}{\|R_{q_k}(A)^{-1}\phi_k\|} = \|R_{q_k}(A)^{-1}\phi_k\|^{-1}.$$

Now, from (5.1) and as $q_k \rightarrow q$, we have

$$\begin{aligned} \|R_q(A)\psi_k\| &= \|(A^2 - 2\text{Re}(q)A + |q|^2 \mathbb{I}_{V_{\mathbb{H}}^R})\psi_k\| \\ &= \|(A^2 - 2\text{Re}(q_k)A + |q_k|^2 \mathbb{I}_{V_{\mathbb{H}}^R} + 2\text{Re}(q_k)A - |q_k|^2 \mathbb{I}_{V_{\mathbb{H}}^R} - 2\text{Re}(q)A + |q|^2 \mathbb{I}_{V_{\mathbb{H}}^R})\psi_k\| \\ &= \|(R_{q_k}(A) - (|q_k|^2 - |q|^2) \mathbb{I}_{V_{\mathbb{H}}^R} + 2(\text{Re}(q_k) - \text{Re}(q))A)\psi_k\| \\ &\leq \|(R_{q_k}(A)\psi_k\| + \||q_k|^2 - |q|^2\| + 2|\text{Re}(q_k) - \text{Re}(q)| \|A\| \\ &= \|R_{q_k}(A)^{-1}\phi_k\|^{-1} + \||q_k|^2 - |q|^2\| + 2|\text{Re}(q_k) - \text{Re}(q)| \|A\| \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

That is, $\|R_q(A)\psi_k\| \rightarrow 0$ as $k \rightarrow \infty$. Hence $q \in \sigma_{ap}^S(A)$ and therefore $\partial\sigma_S(A) \subseteq \sigma_{ap}^S(A)$. Furthermore, from Remark 3.17, $\partial\sigma_S(A) \neq \emptyset$. Thus $\sigma_{ap}^S(A) \neq \emptyset$.

Finally, take an arbitrary $q \in \mathbb{H} \setminus \sigma_{ap}^S(A)$ such that $R_q(A)$ is bounded below. Then, for all $\phi \in \mathcal{D}(A^2)$ and $p \in \mathbb{H}$, there exists $\alpha > 0$ ($\alpha \in \mathbb{R}$) such that

$$\begin{aligned} \alpha\|\phi\| &\leq \|R_q(A)\phi\| \\ &\leq \|R_p(A)\phi\| + \||p|^2 - |q|^2\|\|\phi\| + 2|(\text{Re}(p) - \text{Re}(q))A\phi\|, \end{aligned}$$

which implies

$$(\alpha - \||p|^2 - |q|^2\| - 2|(\text{Re}(p) - \text{Re}(q))A\|)\|\phi\| \leq \|R_p(A)\phi\|.$$

Hence $R_p(A)$ is bounded below for each $p \in \mathbb{H}$ that satisfies

$$\alpha > \||p|^2 - |q|^2\| + 2|(\text{Re}(p) - \text{Re}(q))A\|.$$

From this, we can say that $p \in \mathbb{H} \setminus \sigma_{ap}^S(A)$ for every p which is sufficiently close to q . Hence, there is an open ball $B_\alpha(q)$ centered at q with radius α such that $B_\alpha(q) \subseteq \mathbb{H} \setminus \sigma_{ap}^S(A)$. Therefore $\mathbb{H} \setminus \sigma_{ap}^S(A)$ is open and $\sigma_{ap}^S(A)$ is closed. □

Theorem 5.5. *Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $q \in \mathbb{H}$, and then the following statements are equivalent:*

- (a) $q \notin \sigma_{ap}^S(A)$.
- (b) $q \notin \sigma_l^S(A)$.
- (c) $\bar{q} \notin \sigma_r^S(A^\dagger)$.
- (d) $\text{ran}(R_{\bar{q}}(A^\dagger)) = V_{\mathbb{H}}^R$.

Proof. For $B \in \mathcal{B}(V_{\mathbb{H}}^R)$, we have $BR_q(A) = \mathbb{I}_{V_{\mathbb{H}}^R}$ if and only if $R_{\bar{q}}(A^\dagger)B^\dagger = \mathbb{I}_{V_{\mathbb{H}}^R}$. Therefore (b) and (c) are equivalent.

(a) \Rightarrow (b): Suppose $q \notin \sigma_{ap}^S(A)$. Then by Proposition 5.3, $\ker(\mathbb{R}_q(A)) = \{0\}$. Hence $R_q(A)$ is invertible, and therefore $R_q(A)$ is left invertible. Thus $q \notin \sigma_l^S(A)$.

(c) \Rightarrow (d): Since $\bar{q} \notin \sigma_r^S(A^\dagger)$, $R_{\bar{q}}(A^\dagger)$ is right invertible. Therefore, there is an operator $C \in \mathcal{B}(V_{\mathbb{H}}^R)$ such that $R_{\bar{q}}(A^\dagger)C = \mathbb{I}_{V_{\mathbb{H}}^R}$. Hence, $V_{\mathbb{H}}^R = R_{\bar{q}}(A^\dagger)C(V_{\mathbb{H}}^R) \subseteq \text{ran}(R_{\bar{q}}(A^\dagger))$. Therefore $V_{\mathbb{H}}^R = \text{ran}(R_{\bar{q}}(A^\dagger))$.

(d) \Rightarrow (a): Suppose $\text{ran}(R_{\bar{q}}(A^\dagger)) = V_{\mathbb{H}}^R$. Define $T : \ker(R_{\bar{q}}(A^\dagger))^\perp \rightarrow V_{\mathbb{H}}^R$ by $T\phi = R_{\bar{q}}(A^\dagger)\phi$. Then $\ker(T) = \{0\}$, that is, T is bijective and hence invertible. Let

$$C : V_{\mathbb{H}}^R \rightarrow V_{\mathbb{H}}^R \text{ by } C\phi = T^{-1}\phi.$$

Then $CV_{\mathbb{H}}^R = \ker(R_{\bar{q}}(A^\dagger))^\perp$ and $R_{\bar{q}}(A^\dagger)C = \mathbb{I}_{V_{\mathbb{H}}^R}$. Hence, $C^\dagger R_q(A) = \mathbb{I}_{V_{\mathbb{H}}^R}$ and, for $\phi \in V_{\mathbb{H}}^R$,

$$\|\phi\| = \|C^\dagger R_q(A)\phi\| \leq \|C^\dagger\| \|R_q(A)\phi\|,$$

and from this, we get $\inf\{\|R_q(A)\phi\| \mid \|\phi\| = 1\} \geq \|C^\dagger\|^{-1}$. Therefore, there is no sequence $\{\phi_n\} \subseteq V_{\mathbb{H}}^R$ with $\|\phi_n\| = 1$ and $\|R_q(A)\phi_n\| \rightarrow 0$. Hence $q \notin \sigma_{ap}^S(A)$. □

Corollary 5.6. If $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, then $\partial\sigma_S(A) \subseteq \sigma_l^S(A) \cap \sigma_r^S(A) = \sigma_{ap}^S(A) \cap \sigma_{ap}^S(A^\dagger)^*$.

Proof. From Theorem 5.5, we have $q \notin \sigma_{ap}^S(A) \Leftrightarrow q \notin \sigma_l^S(A)$, and hence $\sigma_{ap}^S(A) = \sigma_l^S(A)$. Also $q \notin \sigma_{ap}^S(A^\dagger) \Leftrightarrow \bar{q} \notin \sigma_r^S(A) \Leftrightarrow q \notin \sigma_r^S(A)^*$, and therefore $\sigma_{ap}^S(A^\dagger)^* = \sigma_r^S(A)$. Let $q \in \partial\sigma_S(A)$, and then by Theorem 5.4 $q \in \sigma_{ap}^S(A) = \sigma_l^S(A)$. If $\bar{q} \notin \sigma_{ap}^S(A^\dagger)$, then by Theorem 5.5 $\text{ran}(R_{\bar{q}}(A^\dagger)) = V_{\mathbb{H}}^R$ and by Proposition 5.3 $\ker(R_{\bar{q}}(A^\dagger)) = \{0\}$. Thus $R_{\bar{q}}(A^\dagger)$ is invertible. Therefore $R_q(A)$ is invertible as $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, and hence $q \notin \sigma_S(A)$, which is a contradiction. Therefore, $\bar{q} \in \sigma_{ap}^S(A^\dagger)$, which implies $q \in \sigma_{ap}^S(A^\dagger)^* = \sigma_r^S(A)$. Thus $q \in \sigma_l^S(A) \cap \sigma_r^S(A)$. \square

Following the complex formalism in the following, we define the S-compression spectrum for an operator $A \in \mathcal{B}(V_{\mathbb{H}}^R)$.

Definition 5.7. The spherical compression spectrum of an operator $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, denoted by $\sigma_c^S(A)$, is defined as

$$\sigma_c^S(A) = \{q \in \mathbb{H} \mid \text{ran}(R_q(A)) \text{ is not dense in } V_{\mathbb{H}}^R\}.$$

Proposition 5.8. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $q \in \mathbb{H}$. Then,

- (a) $q \in \sigma_c^S(A)$ if and only if $\bar{q} \in \sigma_{pS}(A)$;
- (b) $\sigma_c^S(A) \subseteq \sigma_r^S(A)$;
- (c) $\sigma_S(A) = \sigma_{ap}^S(A) \cup \sigma_c^S(A)$.

Proof. (a) $q \in \sigma_c^S(A) \Leftrightarrow \text{ran}(R_q(A)) = \ker(R_{\bar{q}}(A^\dagger))^\perp$ is not dense in $V_{\mathbb{H}}^R \Leftrightarrow \ker(R_{\bar{q}}(A^\dagger)) \neq \{0\} \Leftrightarrow \bar{q} \in \sigma_{pS}(A^\dagger)$.

(b) Let $q \in \sigma_c^S(A)$, and then $\text{ran}(R_q(A))$ is not dense in $V_{\mathbb{H}}^R$. If $q \notin \sigma_r^S(A)$, then by Theorem 5.5 $\bar{q} \notin \sigma_{ap}^S(A^\dagger)$, and thus, again by Theorem 5.5, $\text{ran}(R_{\bar{q}}(A^\dagger)) = V_{\mathbb{H}}^R$, which is contradiction. Therefore $q \in \sigma_r^S(A)$ and hence $\sigma_c^S(A) \subseteq \sigma_r^S(A)$.

(c) Let $q \in \sigma_S(A)$, and then $R_q(A)$ is not invertible in $\mathcal{B}(V_{\mathbb{H}}^R)$. That is, $\ker(R_q(A)) \neq \{0\}$ or $\text{ran}(R_q(A)) \neq V_{\mathbb{H}}^R$. Therefore, by Proposition 5.3, $q \in \sigma_{ap}^S(A)$. Hence $\sigma_S(A) \subseteq \sigma_{ap}^S(A) \cup \sigma_c^S(A)$.

Let $q \in \sigma_{ap}^S(A) \cup \sigma_c^S(A)$. If $q \in \sigma_{ap}^S(A)$, then by Proposition 5.3 $\ker(R_q(A)) \neq \{0\}$ or $\text{ran}(R_q(A))$ is not closed. Hence, $\ker(R_q(A)) \neq \{0\}$ or $\text{ran}(R_q(A)) \neq V_{\mathbb{H}}^R$. Thus $q \in \sigma_S(A)$. If $q \in \sigma_c^S(A)$, then $\text{ran}(R_q(A))$ is not dense in $V_{\mathbb{H}}^R$. Therefore $\text{ran}(R_q(A)) \neq V_{\mathbb{H}}^R$ and hence $q \in \sigma_S(A)$. In summary, $\sigma_{ap}^S(A) \cup \sigma_c^S(A) \subseteq \sigma_S(A)$. Hence (c) holds. \square

VI. FREDHOLM OPERATORS IN THE QUATERNIONIC SETTING

In the complex setting, the Fredholm operators are studied in Banach spaces and Hilbert spaces for bounded and even unbounded linear operators. In this section, we shall study the theory for quaternionic bounded linear operators in separable Hilbert spaces. In this regard, let $V_{\mathbb{H}}^R$ and $U_{\mathbb{H}}^R$ be two separable right quaternionic Hilbert spaces.

Definition 6.1. A Fredholm operator is an operator $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ such that $\ker(A)$ and $\text{coker}(A) = U_{\mathbb{H}}^R/\text{ran}(A)$ are finite dimensional. The dimension of the cokernel is called the codimension, and it is denoted by $\text{codim}(A)$.

Proposition 6.2. If $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ is a Fredholm operator, then $\text{ran}(A)$ is closed.

Proof. Restrict A to $\ker(A)^\perp$, that is, $A|_{\ker(A)^\perp} : \ker(A)^\perp \rightarrow U_{\mathbb{H}}^R$. Then we can assume that $\ker(A) = \{0\}$. Since A is Fredholm, assume that $\text{codim}(A) = \dim(U_{\mathbb{H}}^R/\text{ran}(A)) < \infty$. Therefore, there exists a finite dimensional subspace $V \subseteq U_{\mathbb{H}}^R$ such that $U_{\mathbb{H}}^R = \text{ran}(A) \oplus V$. Since V is finite dimensional, V is closed. Hence $U_{\mathbb{H}}^R = V \oplus V^\perp$. Let $P : U_{\mathbb{H}}^R \rightarrow V^\perp$ be the orthogonal projection onto V^\perp . Hence P is a bounded bijection. Let $G := PA : \ker(A)^\perp \rightarrow V^\perp$, which is a composition of two bounded operators,

and so G is a bounded bijection as A and P are bounded bijections and thereby continuous. Therefore by the open mapping theorem, $G^{-1}: V^\perp \rightarrow \ker(A)^\perp$ is bounded.

Let $\psi \in \text{ran}(A)$. Then, there exists a sequence $\{\phi_n\} \subset V_{\mathbb{H}}^R$ such that $\lim_n A(\phi_n) = \psi$. Thus $\lim_n PA(\phi_n) = P(\psi)$ exists in V^\perp . That is, $\lim_n G(\phi_n) = P(\psi)$ exists in V^\perp . Hence, $\phi := \lim_n \phi_n = G^{-1}P(\psi) = A^{-1}(\psi)$ exists in $V_{\mathbb{H}}^R$. Therefore, $\psi = A(\phi) \in \text{ran}(A)$. Hence $\text{ran}(A)$ is closed. \square

Definition 6.3. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ be a Fredholm operator. Then the index of A is the integer, $\text{ind}(A) = \dim(\ker(A)) - \dim(\text{coker}(A))$.

Remark 6.4. Since $\text{ran}(A)$ is closed, we have $U_{\mathbb{H}}^R = \text{ran}(A) \oplus \text{ran}(A)^\perp = \text{ran}(A) \oplus \ker(A^\dagger)$. Therefore, $\text{coker}(A) = U_{\mathbb{H}}^R / \text{ran}(A) \cong \ker(A^\dagger)$. Thus,

$$\text{ind}(A) = \dim(\ker(A)) - \dim(\ker(A^\dagger)).$$

Theorem 6.5. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ be bijective, and let $K \in \mathcal{B}_0(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ be compact. Then $A + K$ is a Fredholm operator.

Proof. Since A is bijective, A^{-1} exists. Since the kernel of a bounded right linear operator is closed, $\ker(A + K)$ is a Hilbert space. Thus, for $\phi \in \ker(A + K)$, we have $A\phi = -K\phi$. Let $\{\phi_n\}_{n=1}^\infty \subseteq \ker(A + K)$ be a bounded sequence. Since K is a compact operator, $\{K\phi_n\}_{n=1}^\infty$ has a convergence subsequence, $\{K\phi_{n_k}\}_{k=1}^\infty$. Since, for all k , $\phi_{n_k} \in \ker(A + K)$, we have

$$\{K\phi_{n_k}\}_{k=1}^\infty = \{-A\phi_{n_k}\}_{k=1}^\infty.$$

Therefore, $\lim_{k \rightarrow \infty} A\phi_{n_k} = \psi$ exists. Hence, as A^{-1} is continuous, we get $\lim_{k \rightarrow \infty} \phi_{n_k} = A^{-1}\psi$. That is, any bounded sequence in $\ker(A + K)$ has a convergence subsequence. Since an infinite dimensional Hilbert space has an infinite orthonormal sequence with no convergence subsequence, $\dim(\ker(A + K)) < \infty$.

Since A is invertible and K is compact, A^\dagger is invertible and K^\dagger is compact. Therefore, by the above argument, $\dim(\ker(A^\dagger + K^\dagger)) < \infty$. Thus, since $U_{\mathbb{H}}^R = \text{ran}(A + K) \oplus \ker(A^\dagger + K^\dagger)$, to show that $\text{codim}(A + K) < \infty$, it is enough to show that $\text{ran}(A + K)$ is closed. Split $V_{\mathbb{H}}^R$ as $V_{\mathbb{H}}^R = \mathfrak{H} \oplus \ker(A + K)$, where \mathfrak{H} is a subspace of $V_{\mathbb{H}}^R$ such that $\mathfrak{H} \perp \ker(A + K)$, and consider the restriction of $A + K$ to \mathfrak{H} .

Claim. For $\phi \in \mathfrak{H}$, the inequality

$$\|\phi\|_{V_{\mathbb{H}}^R} \leq c \|(A + K)\phi\|_{U_{\mathbb{H}}^R}$$

holds for some $c > 0$.

If for all $c > 0$ there exists $\phi \in \mathfrak{H}$ such that $\|\phi\|_{V_{\mathbb{H}}^R} > c \|(A + K)\phi\|_{U_{\mathbb{H}}^R}$, then there exist sequences $\{c_n\}_{n=1}^\infty \subseteq (0, \infty)$ and $\{\phi_n\}_{n=1}^\infty \subseteq \mathfrak{H}$ with $\|\phi_n\|_{V_{\mathbb{H}}^R} = 1$ for all n such that $c_n \rightarrow \infty$ as $n \rightarrow \infty$. Furthermore, for all n ,

$$1 = \|\phi_n\|_{V_{\mathbb{H}}^R} > c_n \|(A + K)\phi_n\|_{U_{\mathbb{H}}^R}.$$

Thus $\|(A + K)\phi_n\|_{U_{\mathbb{H}}^R} < \frac{1}{c_n} \rightarrow 0$ as $n \rightarrow \infty$. Since K is compact and, for each n , $\|\phi_n\|_{V_{\mathbb{H}}^R} = 1$, there exists a convergent subsequence $\{K\phi_{n_k}\}_{k=1}^\infty$ of $\{K\phi_n\}_{n=1}^\infty$. Assume $K\phi_{n_k} \rightarrow \psi \in U_{\mathbb{H}}^R$ as $k \rightarrow \infty$. Then, $A\phi_{n_k} \rightarrow -\psi \in U_{\mathbb{H}}^R$ as $k \rightarrow \infty$. Hence $\phi_{n_k} = A^{-1}A\phi_{n_k} \rightarrow \eta = -A^{-1}\psi$ as $k \rightarrow \infty$, where $\eta \in \mathfrak{H}$ and $\|\eta\|_{V_{\mathbb{H}}^R} = 1$ as $\|\phi_{n_k}\|_{V_{\mathbb{H}}^R} = 1$ for all k . But $(A + K)\eta = \lim_{k \rightarrow \infty} (A\phi_{n_k} + K\phi_{n_k}) = \lim_{k \rightarrow \infty} 0 = 0$. Therefore $\eta \in \ker(A + K)$, which is a contradiction to $\mathfrak{H} \perp \ker(A + K)$. Hence the claim holds. Thus, by Proposition 3.5, $\text{ran}(A + K)$ is closed. Hence $\text{codim}(A + K) < \infty$ and $A + K$ is Fredholm. \square

Proposition 6.6. If $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ is Fredholm, then $A^\dagger \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ is Fredholm.

Proof. Since A is Fredholm, $\dim(\ker(A)) < \infty$ and $\dim(U_{\mathbb{H}}^R/\text{ran}(A)) < \infty$. Then by Proposition 6.2, $\text{ran}(A)$ is closed. Therefore, $U_{\mathbb{H}}^R = \text{ran}(A) \oplus \text{ran}(A)^\perp = \text{ran}(A) \oplus \ker(A^\dagger)$. Thus, $U_{\mathbb{H}}^R/\text{ran}(A) \cong \ker(A^\dagger)$. Hence $\dim(\ker(A^\dagger)) < \infty$. Also, since by Proposition 3.4, $\ker(A)$ is closed, we have $\text{coker}(A^\dagger) = V_{\mathbb{H}}^R/\text{ran}(A^\dagger) = V_{\mathbb{H}}^R/\ker(A)^\perp \cong \ker(A)$. Thus $\dim(\text{coker}(A^\dagger)) = \dim(\ker(A)) < \infty$. Hence A^\dagger is Fredholm. \square

Remark 6.7. From Propositions 6.2 and 6.6, $\text{ran}(A^\dagger)$ is closed. Therefore,

$$\text{ind}(A^\dagger) = \dim(\ker(A^\dagger)) - \dim(\ker((A^\dagger)^\dagger)) = \dim(\ker(A^\dagger)) - \dim(\ker(A)) = -\text{ind}(A).$$

Theorem 6.8. $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ is Fredholm if and only if there exist $S_1, S_2 \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ and compact operators K_1 and K_2 , on $V_{\mathbb{H}}^R$ and $U_{\mathbb{H}}^R$, respectively, such that

$$S_1A = \mathbb{I}_{V_{\mathbb{H}}^R} + K_1 \quad \text{and} \quad AS_2 = \mathbb{I}_{U_{\mathbb{H}}^R} + K_2.$$

Proof. (\Rightarrow) Suppose $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ is a Fredholm operator. Then A defines a bijective operator

$$\hat{A} : \mathfrak{H}_1 = (\ker(A))^\perp \longrightarrow \mathfrak{H}_2 = \text{ran}(A) = (\ker(A^\dagger))^\perp.$$

Then $\hat{A}^{-1} : \mathfrak{H}_2 \longrightarrow \mathfrak{H}_1$ exists and is bounded. Consider the orthogonal projection operator $P_r : U_{\mathbb{H}}^R \longrightarrow \mathfrak{H}_2$ and the inclusion operator $i : \mathfrak{H}_1 \longrightarrow V_{\mathbb{H}}^R$. Let $S_2 = i\hat{A}^{-1}P_r : U_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$, and then $S_2 \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ and $AS_2 = A i \hat{A}^{-1} P_r = \mathbb{I}_{U_{\mathbb{H}}^R} - P_a$, where P_a is the orthogonal projection operator $P_a : U_{\mathbb{H}}^R \longrightarrow \ker(A^\dagger)$. Let $K_2 = -P_a$. Since $\ker(A^\dagger)$ is finite dimensional, K_2 is a finite rank operator, so it is compact. That is, $AS_2 = \mathbb{I}_{U_{\mathbb{H}}^R} + K_2$.

From Proposition 6.6, A^\dagger is Fredholm. Therefore, from the above argument, there exist $S_3 \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ and a compact operator on $V_{\mathbb{H}}^R$ such that $A^\dagger S_3 = \mathbb{I}_{V_{\mathbb{H}}^R} + K_3$. Hence, $S_3^\dagger A = \mathbb{I}_{V_{\mathbb{H}}^R} + K_3^\dagger$. Set $S_1 = S_3^\dagger$ and $K_1 = K_3^\dagger$, then $S_1 \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$, K_1 is compact on $V_{\mathbb{H}}^R$, and $S_1A = \mathbb{I}_{V_{\mathbb{H}}^R} + K_1$.

(\Leftarrow) Suppose that there are operators $S_1, S_2 \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ and compact operators K_1 on $V_{\mathbb{H}}^R$ and K_2 on $U_{\mathbb{H}}^R$ such that $S_1A = \mathbb{I}_{V_{\mathbb{H}}^R} + K_1$ and $AS_2 = \mathbb{I}_{U_{\mathbb{H}}^R} + K_2$. We have the obvious inclusions,

$$\ker(A) \subseteq \ker(S_1A) = \ker(\mathbb{I}_{V_{\mathbb{H}}^R} + K_1), \tag{6.1}$$

$$\text{ran}(A) \supseteq \text{ran}(AS_2) = \text{ran}(\mathbb{I}_{U_{\mathbb{H}}^R} + K_2). \tag{6.2}$$

By Theorem 6.5, $\mathbb{I}_{V_{\mathbb{H}}^R} + K_1$ and $\mathbb{I}_{U_{\mathbb{H}}^R} + K_2$ are Fredholm operators. Therefore from Eq. (6.1), $\dim(\ker(A)) \leq \dim(\mathbb{I}_{V_{\mathbb{H}}^R} + K_1) < \infty$, and by Eq. (6.2), $\text{codim}(A) \leq \text{codim}(\mathbb{I}_{U_{\mathbb{H}}^R} + K_2) < \infty$. Hence A is a Fredholm operator. \square

Remark 6.9. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$, and then

- (a) A is said to be left semi-Fredholm if there exist $B \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ and a compact operator K_1 on $V_{\mathbb{H}}^R$ such that $BA = \mathbb{I}_{V_{\mathbb{H}}^R} + K_1$, and the set of all left semi-Fredholm operators are denoted by $\mathcal{F}_l(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$; ⁸
- (b) A is said to be right semi-Fredholm if there exist $B \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ and a compact operator K_2 on $U_{\mathbb{H}}^R$ such that $AB = \mathbb{I}_{U_{\mathbb{H}}^R} + K_2$, and the set of all right semi-Fredholm operators are denoted by $\mathcal{F}_r(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$; ⁸
- (c) by Theorem 6.8, the set of all Fredholm operators $\mathcal{F}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R) = \mathcal{F}_l(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R) \cap \mathcal{F}_r(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$;
- (d) from Theorem 6.8, it is also clear that every invertible right linear operator is Fredholm;
- (e) let $\mathcal{SF}(V_{\mathbb{H}}^R) = \mathcal{F}_l(V_{\mathbb{H}}^R) \cup \mathcal{F}_r(V_{\mathbb{H}}^R)$, and from Theorems 6.8 and 4.9, we have

$$\begin{aligned} A \in \mathcal{F}_l(V_{\mathbb{H}}^R) &\Leftrightarrow A^\dagger \in \mathcal{F}_r(V_{\mathbb{H}}^R), \\ A \in \mathcal{SF}(V_{\mathbb{H}}^R) &\Leftrightarrow A^\dagger \in \mathcal{SF}(V_{\mathbb{H}}^R), \\ A \in \mathcal{F}(V_{\mathbb{H}}^R) &\Leftrightarrow A^\dagger \in \mathcal{F}(V_{\mathbb{H}}^R). \end{aligned}$$

Proposition 6.10. $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ is Fredholm if and only if there exists $S \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ such that $AS - \mathbb{I}_{U_{\mathbb{H}}^R}$ and $SA - \mathbb{I}_{V_{\mathbb{H}}^R}$ are both finite rank operators.

Proof. (\Rightarrow) Suppose A is a Fredholm operator, then $\text{ran}(A)$ is closed and $A: \ker(A)^\perp \rightarrow \text{ran}(A)$ is a bijection between Hilbert spaces. Let \hat{A} be the inverse of this bijection. By the open mapping theorem, $\hat{A} \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$. Let $P: U_{\mathbb{H}}^R \rightarrow \text{ran}(A)$ be the orthogonal projection onto $\text{ran}(A)$. Set $S = \hat{A}P$ and then

$$SA - \mathbb{I}_{V_{\mathbb{H}}^R} = \hat{A}PA - \mathbb{I}_{V_{\mathbb{H}}^R} = \hat{A}A - \mathbb{I}_{V_{\mathbb{H}}^R} = -P_k,$$

where P_k is the orthogonal projection onto $\ker(A)$, hence a finite rank operator. Also

$$AS - \mathbb{I}_{U_{\mathbb{H}}^R} = A\hat{A}P - \mathbb{I}_{U_{\mathbb{H}}^R} = -(\mathbb{I}_{U_{\mathbb{H}}^R} - P),$$

which is also a finite rank operator [because $\text{ran}(A)^\perp = \ker(A^\dagger)$ is finite dimensional as A^\dagger is Fredholm]. Therefore, $SA - \mathbb{I}_{V_{\mathbb{H}}^R}$ and $AS - \mathbb{I}_{U_{\mathbb{H}}^R}$ are finite rank operators and hence compact.

(\Leftarrow) Now suppose that $SA - \mathbb{I}_{V_{\mathbb{H}}^R} = F_1$ and $AS - \mathbb{I}_{U_{\mathbb{H}}^R} = F_2$ with F_1 and F_2 being finite rank operators on $V_{\mathbb{H}}^R$ and $U_{\mathbb{H}}^R$, respectively. By Proposition 4.6, F_1 and F_2 are compact operators. Therefore by Theorem 6.8, A is a Fredholm operator. \square

Definition 6.11. Let V_0, \dots, V_n be right quaternionic vector spaces, and let $A_j: V_j \rightarrow V_{j+1}$, $0 \leq j \leq n-1$, be right linear operators. Then the sequence

$$V_0 \xrightarrow{A_0} V_1 \xrightarrow{A_1} V_2 \xrightarrow{A_2} \dots \xrightarrow{A_{n-2}} V_{n-1} \xrightarrow{A_{n-1}} V_n$$

is called exact if $\text{ran}(A_j) = \ker(A_{j+1})$ for $j = 0, 1, \dots, n-2$.

Lemma 6.12. Let V_j be a finite dimensional right quaternionic vector space for each $j = 0, 1, \dots, n$. If

$$\{0\} = V_0 \xrightarrow{A_0} V_1 \xrightarrow{A_1} V_2 \xrightarrow{A_2} \dots \xrightarrow{A_{n-2}} V_{n-1} \xrightarrow{A_{n-1}} V_n = \{0\}$$

be an exact sequence with $\dim(V_j) < \infty$ for all $j = 0, 1, \dots, n$, then

$$\sum_{j=0}^{n-1} (-1)^j \dim(V_j) = 0.$$

Proof. For each j , decompose V_j as $V_j = \ker(A_j) \oplus Y_j$, where Y_j is an orthogonal complement of $\ker(A_j)$. The exactness of the sequence implies that $A_j: Y_j \rightarrow \ker(A_{j+1}) = \text{ran}(A_j)$ is an isomorphism. Therefore, $\dim(Y_j) = \dim(\ker(A_{j+1}))$ for each $j = 0, 1, \dots, n-1$. Hence $\dim(V_j) = \dim(\ker(A_j)) + \dim(\ker(A_{j+1}))$. Furthermore, $\dim(\ker(A_0)) = 0$ and $\dim(V_{n-1}) = \dim(\ker(A_{n-1}))$. Hence the result follows. \square

Theorem 6.13. Let $V_{\mathbb{H}}^R, U_{\mathbb{H}}^R$ and $W_{\mathbb{H}}^R$ be right quaternionic Hilbert spaces. If $A_1 \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ and $A_2 \in \mathcal{B}(U_{\mathbb{H}}^R, W_{\mathbb{H}}^R)$ are two Fredholm operators, then $A_2A_1 \in \mathcal{B}(V_{\mathbb{H}}^R, W_{\mathbb{H}}^R)$ is also a Fredholm operator, and it satisfies $\text{ind}(A_2A_1) = \text{ind}(A_1) + \text{ind}(A_2)$.

Proof. Suppose that A_1 and A_2 be Fredholm operators. Clearly, $\ker(A_1) \subseteq \ker(A_2A_1)$ and $A_1: \ker(A_2A_1) \rightarrow \ker(A_2)$. Therefore, there is an isomorphism

$$S: \ker(A_2A_1)/\ker(A_1) \rightarrow B \subseteq \ker(A_2),$$

where B is a subspace of $\ker(A_2)$. Hence, $\dim(\ker(A_2A_1)/\ker(A_1)) \leq \dim(\ker(A_2)) < \infty$, and thus

$$\dim(\ker(A_2A_1)) \leq \dim(\ker(A_1)) + \dim(\ker(A_2)) < \infty.$$

Since $\text{ran}(A_2A_1) \subseteq \text{ran}(A_2) \subseteq W_{\mathbb{H}}^R$, we have

$$\left(W_{\mathbb{H}}^R / \text{ran}(A_2A_1) \right) / (\text{ran}(A_2) / \text{ran}(A_2A_1)) = W_{\mathbb{H}}^R / \text{ran}(A_2).$$

Hence,

$$\dim(\text{coker}(A_2A_1)) = \dim(\text{ran}(A_2) / \text{ran}(A_2A_1)) + \dim(\text{coker}(A_2)). \tag{6.3}$$

Furthermore, $A_2 : U_{\mathbb{H}}^R / \text{ran}(A_1) \rightarrow \text{ran}(A_2) / \text{ran}(A_2A_1)$ is surjective, and therefore

$$\dim(\text{ran}(A_2) / \text{ran}(A_2A_1)) \leq \dim(\text{coker}(A_1)) < \infty. \tag{6.4}$$

From Eqs. (6.3) and (6.4), we have

$$\dim(\text{coker}(A_2A_1)) \leq \dim(\text{coker}(A_1)) + \dim(\text{coker}(A_2)) < \infty.$$

Thus A_2A_1 is a Fredholm operator.

Let $i : \ker(A_1) \rightarrow \ker(A_2A_1)$ be the inclusion map, $Q : \ker(A_2) \rightarrow U_{\mathbb{H}}^R / \text{ran}(A_1)$ be the quotient map, and $E : W_{\mathbb{H}}^R / \text{ran}(A_2A_1) \rightarrow W_{\mathbb{H}}^R / \text{ran}(A_2)$ map equivalence classes modulo $\text{ran}(A_2A_1)$ to equivalence classes modulo $\text{ran}(A_2)$. Consider the sequence

$$\begin{aligned} \{0\} \xrightarrow{O_1} \ker(A_1) \xrightarrow{i} \ker(A_2A_1) \xrightarrow{A_1} \ker(A_2) \xrightarrow{Q} U_{\mathbb{H}}^R / \text{ran}(A_1) \rightarrow \\ \xrightarrow{A_2} W_{\mathbb{H}}^R / \text{ran}(A_2A_1) \xrightarrow{E} W_{\mathbb{H}}^R / \text{ran}(A_2) \xrightarrow{O_2} \{0\}. \end{aligned}$$

Since $\text{ran}(O_1) = \ker(i) = \{0\}$, $\text{ran}(i) = \ker(A_1)$, $\text{ran}(A_1) = \ker(Q)$, $\text{ran}(Q) = \ker(A_2)$, and $\text{ran}(A_2) = \ker(E)$, the above sequence is exact. Therefore, by Lemma 6.12, we have

$$\begin{aligned} 0 &= \dim(\{0\}) - \dim(\ker(A_1)) + \dim(\ker(A_2A_1)) - \dim(\ker(A_2)) \\ &\quad + \dim\left(U_{\mathbb{H}}^R / \text{ran}(A_1) \right) - \dim\left(W_{\mathbb{H}}^R / \text{ran}(A_2A_1) \right) + \dim\left(W_{\mathbb{H}}^R / \text{ran}(A_2) \right) \\ &= [\dim(\ker(A_2A_1)) - \dim\left(W_{\mathbb{H}}^R / \text{ran}(A_2A_1) \right)] - [\dim(\ker(A_1)) - \dim\left(U_{\mathbb{H}}^R / \text{ran}(A_1) \right)] \\ &\quad - [\dim(\ker(A_2)) - \dim\left(W_{\mathbb{H}}^R / \text{ran}(A_2) \right)] \\ &= \text{ind}(A_2A_1) - \text{ind}(A_1) - \text{ind}(A_2). \end{aligned}$$

Hence the result follows. □

Corollary 6.14. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ and $B \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ such that $AB = F$ and $BA = G$ are Fredholm operators. Then A and B are Fredholm operators and $\text{ind}(AB) = \text{ind}(A) + \text{ind}(B)$.

Proof. Since $\text{ran}(F) \subseteq \text{ran}(A)$ and $\ker(A) \subseteq \ker(G)$, we have $\dim(\ker(A)) < \infty$ and $\dim(U_{\mathbb{H}}^R / \text{ran}(A)) \leq \dim(U_{\mathbb{H}}^R / \text{ran}(F)) < \infty$. Hence A is Fredholm. Similarly B is also Fredholm. The index equality follows from Theorem 6.13. □

Lemma 6.15. Let $F \in \mathcal{B}(V_{\mathbb{H}}^R)$ be a finite rank operator, and then $\text{ind}(\mathbb{I}_{V_{\mathbb{H}}^R} + F) = 0$.

Proof. Suppose that $F \in \mathcal{B}(V_{\mathbb{H}}^R)$ be a finite rank operator. Then, $\dim(\text{ran}(F)) < \infty$. Also by Theorem 4.6, $\ker(F)^\perp = \text{ran}(F^\dagger)$ is also a finite dimensional subspace. Define $L := \text{ran}(F) + \ker(F)^\perp$ with $\dim(L) < \infty$, and hence L is closed. Thus $V_{\mathbb{H}}^R = L \oplus L^\perp$. Furthermore,

$$\begin{aligned} (\mathbb{I}_{V_{\mathbb{H}}^R} + F)L &\subseteq L + FL \subseteq L, \\ (\mathbb{I}_{V_{\mathbb{H}}^R} + F)|_{L^\perp} &= \mathbb{I}_{L^\perp}. \end{aligned}$$

Hence L and L^\perp are invariant under $\mathbb{I}_{V_{\mathbb{H}}} + F$, and therefore,

$$\text{ind}(\mathbb{I}_{V_{\mathbb{H}}} + F) = \text{ind}((\mathbb{I}_{V_{\mathbb{H}}} + F)|_L) + \text{ind}((\mathbb{I}_{V_{\mathbb{H}}} + F)|_{L^\perp}).$$

Since $\ker((\mathbb{I}_{V_{\mathbb{H}}} + F)|_{L^\perp}) = \{0\}$ and $L^\perp/\text{ran}((\mathbb{I}_{V_{\mathbb{H}}} + F)|_{L^\perp}) \cong \{0\}$, $\text{ind}((\mathbb{I}_{V_{\mathbb{H}}} + F)|_{L^\perp}) = 0$. Furthermore, since $\dim(L) < \infty$, $\dim(L) = \dim(\ker((\mathbb{I}_{V_{\mathbb{H}}} + F)|_L)) + \dim(\text{ran}((\mathbb{I}_{V_{\mathbb{H}}} + F)|_L))$, and therefore $\text{codim}((\mathbb{I}_{V_{\mathbb{H}}} + F)|_L) = \dim(\ker((\mathbb{I}_{V_{\mathbb{H}}} + F)|_L))$. Hence, $\text{ind}(\mathbb{I}_{V_{\mathbb{H}}} + F)|_L = 0$, which concludes the proof. \square

Theorem 6.16. *Let $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ be a Fredholm operator, and then for any compact operator $K \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$, $A + K$ is a Fredholm operator and $\text{ind}(A + K) = \text{ind}(A)$.*

Proof. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ be a Fredholm operator, and then by Theorem 6.8, there exist $S_1, S_2 \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ and compact operators K_1, K_2 on $V_{\mathbb{H}}^R, U_{\mathbb{H}}^R$, respectively, such that $S_1A = \mathbb{I}_{V_{\mathbb{H}}} + K_1$ and $AS_2 = \mathbb{I}_{U_{\mathbb{H}}} + K_2$. Hence we have

$$\begin{aligned} S_1(A + K) &= S_1A + S_1K = \mathbb{I}_{V_{\mathbb{H}}} + K_1 + S_1K = \mathbb{I}_{V_{\mathbb{H}}} + K'_1, \\ (A + K)S_2 &= AS_2 + KS_2 = \mathbb{I}_{U_{\mathbb{H}}} + K_2 + KS_2 = \mathbb{I}_{U_{\mathbb{H}}} + K'_2, \end{aligned}$$

where, by Proposition 4.2, $K'_1 = K_1 + S_1K$ and $K'_2 = K_2 + KS_2$ are compact operators. Therefore by Theorem 6.8, $A + K$ is a Fredholm operator.

Now by Proposition 6.10, there exist $S \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ and finite rank operators F_1 and F_2 , respectively, on $V_{\mathbb{H}}^R$ and $U_{\mathbb{H}}^R$ such that

$$SA = \mathbb{I}_{V_{\mathbb{H}}} + F_1 \quad \text{and} \quad AS = \mathbb{I}_{U_{\mathbb{H}}} + F_2.$$

Hence by Proposition 4.6 and Theorem 6.5, SA and AS are Fredholm operators. Thus by Corollary 6.14, S is a Fredholm operator. Hence by Lemma 6.15 and Theorem 6.13,

$$0 = \text{ind}(SA) = \text{ind}(S) + \text{ind}(A). \tag{6.5}$$

Since S and $A + K$ are Fredholm operators, by Theorem 6.13, $S(A + K)$ is a Fredholm operator. Therefore by Theorem 6.10, there exists a finite rank operator F_3 on $V_{\mathbb{H}}^R$ such that $S(A + K) = \mathbb{I}_{V_{\mathbb{H}}} + F_3$. Thus by Lemma 6.15 and Theorem 6.13, we have

$$0 = \text{ind}(S(A + K)) = \text{ind}(S) + \text{ind}(A + K). \tag{6.6}$$

From Eqs. (6.5) and (6.6), we have $\text{ind}(A + K) = \text{ind}(A)$. \square

Corollary 6.17. *Let $K \in \mathcal{B}(V_{\mathbb{H}}^R)$ be compact, and then $\mathbb{I}_{V_{\mathbb{H}}} + K$ is Fredholm and $\text{ind}(\mathbb{I} + K) = 0$.*

Proof. Let $K \in \mathcal{B}_0(V_{\mathbb{H}}^R)$. Clearly $\mathbb{I}_{V_{\mathbb{H}}}$ is Fredholm and $\text{ind}(\mathbb{I}_{V_{\mathbb{H}}}) = 0$. Therefore, by Theorem 6.16, $\mathbb{I}_{V_{\mathbb{H}}} + K$ is Fredholm and $\text{ind}(\mathbb{I}_{V_{\mathbb{H}}} + K) = \text{ind}(\mathbb{I}_{V_{\mathbb{H}}}) = 0$. \square

Corollary 6.18. *Every invertible operator $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ is Fredholm and $\text{ind}(A) = 0$.*

Proof. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ is invertible. Then $AA^{-1} = A^{-1}A = \mathbb{I}_{V_{\mathbb{H}}}$ is Fredholm. Therefore, by Corollary 6.14, A and A^{-1} are Fredholm operators. Since A is invertible, so is A^\dagger . Therefore, $\ker(A) = \ker(A^\dagger) = \{0\}$. Hence $\text{ind}(A) = 0$. \square

Corollary 6.19. *Let n be a non-negative integer. If $A \in \mathcal{F}(V_{\mathbb{H}}^R)$, and then $A^n \in \mathcal{F}(V_{\mathbb{H}}^R)$ and $\text{ind}(A^n) = n \text{ind}(A)$.*

Proof. The result is trivial for $n = 0, 1$. $n = 2$ follows from Theorem 6.13, and by induction, we have the result. \square

Theorem 6.20. *Let $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ be a Fredholm operator. Then there exists a constant $c > 0$ such that for every operator $S \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ with $\|S\| < c$, $A + S$ is a Fredholm operator and $\text{ind}(A + S) = \text{ind}(A)$.*

Proof. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R, U_{\mathbb{H}}^R)$ be a Fredholm operator, and then by Proposition 6.6, $A^\dagger \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ is Fredholm. Therefore, $\ker(A)$ and $\ker(A^\dagger)$ are finite dimensional subspaces of $V_{\mathbb{H}}^R$ and $U_{\mathbb{H}}^R$, respectively. Let $R_1, R_2 \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ be operators such that $R_1A = \mathbb{I}_{V_{\mathbb{H}}^R} - P_k$ and $AR_2 = \mathbb{I}_{U_{\mathbb{H}}^R} - P_{k^\dagger}$, where $P_k : V_{\mathbb{H}}^R \rightarrow \ker(A)$ and $P_{k^\dagger} : U_{\mathbb{H}}^R \rightarrow \ker(A^\dagger)$ be orthogonal projections. Furthermore, P_k and P_{k^\dagger} are finite rank operators and hence compact. Now $R_1(A + S) = R_1A + R_1S = \mathbb{I}_{V_{\mathbb{H}}^R} + R_1S - P_k$. For $\|S\| < \min\{\|R_1\|^{-1}, \|R_2\|^{-1}\}$, we have $\|R_1S\| < 1$. Therefore by Proposition 3.6, $\mathbb{I}_{V_{\mathbb{H}}^R} + R_1S$ is invertible, and by the open mapping theorem, $(\mathbb{I}_{V_{\mathbb{H}}^R} + R_1S)^{-1} \in \mathcal{B}(V_{\mathbb{H}}^R)$. Now

$$(\mathbb{I}_{V_{\mathbb{H}}^R} + R_1S)^{-1}R_1(A + S) = \mathbb{I}_{V_{\mathbb{H}}^R} - (\mathbb{I}_{V_{\mathbb{H}}^R} + R_1S)^{-1}P_k = \mathbb{I}_{V_{\mathbb{H}}^R} + K_1,$$

where, by Proposition 4.2, $K_1 = -(\mathbb{I}_{V_{\mathbb{H}}^R} + R_1S)^{-1}P_k$ is a compact operator on $V_{\mathbb{H}}^R$. Furthermore, $(A + S)R_2 = AR_2 + SR_2 = \mathbb{I}_{U_{\mathbb{H}}^R} + SR_2 - P_{k^\dagger}$. Again $\|SR_2\| < 1$, and therefore by the same argument as above $\mathbb{I}_{U_{\mathbb{H}}^R} + SR_2$ is invertible and $(\mathbb{I}_{U_{\mathbb{H}}^R} + SR_2)^{-1} \in \mathcal{B}(U_{\mathbb{H}}^R)$. Hence,

$$(A + S)R_2(\mathbb{I}_{U_{\mathbb{H}}^R} + SR_2)^{-1} = \mathbb{I}_{U_{\mathbb{H}}^R} - (\mathbb{I}_{U_{\mathbb{H}}^R} + SR_2)^{-1}P_{k^\dagger} = \mathbb{I}_{U_{\mathbb{H}}^R} + K_2,$$

where $K_2 = -(\mathbb{I}_{U_{\mathbb{H}}^R} + SR_2)^{-1}P_{k^\dagger}$ is a compact operator on $U_{\mathbb{H}}^R$. That is, we have operators $S_1 = (\mathbb{I}_{V_{\mathbb{H}}^R} + R_1S)^{-1}R_1, S_2 = R_2(\mathbb{I}_{U_{\mathbb{H}}^R} + SR_2)^{-1} \in \mathcal{B}(U_{\mathbb{H}}^R, V_{\mathbb{H}}^R)$ and compact operators K_1 and K_2 on $V_{\mathbb{H}}^R$ and $U_{\mathbb{H}}^R$, respectively, such that

$$S_1(A + S) = \mathbb{I}_{V_{\mathbb{H}}^R} + K_1 \quad \text{and} \quad (A + S)S_2 = \mathbb{I}_{U_{\mathbb{H}}^R} + K_2.$$

Therefore, by Theorem 6.8, $A + S$ is a Fredholm operator. The identity, $\text{ind}(A + S) = \text{ind}(A)$, follows similar to the Proof in Theorem 6.16. \square

Theorem 6.21. *An operator $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ is left semi-Fredholm if and only if $\text{ran}(A)$ is closed and $\ker(A)$ is finite dimensional. Hence*

$$\mathcal{F}_l(V_{\mathbb{H}}^R) = \{A \in \mathcal{B}(V_{\mathbb{H}}^R) \mid \text{ran}(A) \text{ is closed and } \dim(\ker(A)) < \infty\}, \tag{6.7}$$

$$\mathcal{F}_r(V_{\mathbb{H}}^R) = \{A \in \mathcal{B}(V_{\mathbb{H}}^R) \mid \text{ran}(A) \text{ is closed and } \dim(\ker(A^\dagger)) < \infty\}. \tag{6.8}$$

Proof. (\Rightarrow) Suppose that $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ is left semi-Fredholm. Then there exist $S \in \mathcal{B}(V_{\mathbb{H}}^R)$ and a compact operator K in $V_{\mathbb{H}}^R$ such that $SA = \mathbb{I}_{V_{\mathbb{H}}^R} + K$. We have $\ker(A) \subseteq \ker(SA) = \ker(\mathbb{I}_{V_{\mathbb{H}}^R} + K)$ and $\text{ran}(SA) = \text{ran}(\mathbb{I}_{V_{\mathbb{H}}^R} + K)$. By Theorem 6.5, $\mathbb{I}_{V_{\mathbb{H}}^R} + K$ is Fredholm and hence $\dim(\ker(\mathbb{I}_{V_{\mathbb{H}}^R} + K)) < \infty$ which implies $\dim(\ker(A)) < \infty$. By Proposition 6.2, $\text{ran}(\mathbb{I}_{V_{\mathbb{H}}^R} + K)$ is closed. That is, $\dim(\ker(A)) < \infty, \dim(A(\ker(SA))) < \infty$, and $\text{ran}(SA)$ is closed. Consider the operator

$$(SA)|_{\ker(SA)^\perp} : \ker(SA)^\perp \rightarrow V_{\mathbb{H}}^R.$$

Since this operator is injective with a closed range, by Theorem 3.7, it is bounded below. Hence, there exists $c > 0$ such that

$$c\|\phi\|_{V_{\mathbb{H}}^R} \leq \|SA\phi\|_{V_{\mathbb{H}}^R} \leq \|S\|\|A\phi\|_{V_{\mathbb{H}}^R} \text{ for all } \phi \in \ker(SA)^\perp.$$

Hence,

$$A|_{\ker(SA)^\perp} : \ker(SA)^\perp \longrightarrow V_{\mathbb{H}}^R$$

is bounded below; therefore by Theorem 3.7, $A|_{\ker(SA)^\perp}$ has a closed range. Since $V_{\mathbb{H}}^R = \ker(SA) \oplus \ker(SA)^\perp$, we have

$$\text{ran}(A) = A(V_{\mathbb{H}}^R) = A(\ker(SA) \oplus \ker(SA)^\perp) = A(\ker(SA)) \oplus A(\ker(SA)^\perp),$$

which is closed because the direct sum of a closed subspace and a finite dimensional space is closed.

(\Leftarrow) Suppose that $\dim(\ker(A)) < \infty$ and $\text{ran}(A)$ is closed. Hence, $A|_{\ker(A)^\perp} : \ker(A)^\perp \longrightarrow V_{\mathbb{H}}^R$ is injective and has a closed range, $\text{ran}(A|_{\ker(A)^\perp}) = \text{ran}(A)$. Therefore by Theorem 3.7, it has a bounded inverse. Let $P_r : V_{\mathbb{H}}^R \longrightarrow \text{ran}(A)$ be the orthogonal projection onto $\text{ran}(A)$. Define $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ by $T = (A|_{\ker(A)^\perp})^{-1}P_r$. If $\phi \in \ker(A)$, then $TA\phi = 0$. If $\psi \in \ker(A)^\perp$, then

$$TA\psi = (A|_{\ker(A)^\perp})^{-1}P_r(A|_{\ker(A)^\perp})\psi = (A|_{\ker(A)^\perp})^{-1}(A|_{\ker(A)^\perp})\psi = \psi.$$

Therefore, for every $\eta = \phi + \psi \in V_{\mathbb{H}}^R = \ker(A) \oplus \ker(A)^\perp$,

$$TA\eta = TA\phi + TA\psi = TA\psi = \psi = P_n\psi,$$

where $P_n : V_{\mathbb{H}}^R \longrightarrow \ker(A)^\perp$ is the orthogonal projection onto $\ker(A)^\perp$. Thus,

$$TA = P_n = \mathbb{I}_{V_{\mathbb{H}}^R} - (\mathbb{I}_{V_{\mathbb{H}}^R} - P_n) = \mathbb{I}_{V_{\mathbb{H}}^R} + K,$$

where $K = -(\mathbb{I}_{V_{\mathbb{H}}^R} - P_n)$ is a finite rank operator as it is an orthogonal projection onto the finite dimensional subspace $\ker(A)$. That is, $TA = \mathbb{I}_{V_{\mathbb{H}}^R} + K$ and K is compact, and hence A is left semi-Fredholm. Therefore,

$$\mathcal{F}_l(V_{\mathbb{H}}^R) = \{A \in \mathcal{B}(V_{\mathbb{H}}^R) \mid \text{ran}(A) \text{ is closed and } \dim(\ker(A)) < \infty\}. \tag{6.9}$$

Now by Remark 6.9, $A^\dagger \in \mathcal{F}_l(V_{\mathbb{H}}^R) \Leftrightarrow A \in \mathcal{F}_r(V_{\mathbb{H}}^R)$, and by Proposition 3.8, $\text{ran}(A)$ is closed if and only if $\text{ran}(A^\dagger)$ is closed. Hence, from Eq. (6.9), we have

$$\mathcal{F}_r(V_{\mathbb{H}}^R) = \{A \in \mathcal{B}(V_{\mathbb{H}}^R) \mid \text{ran}(A) \text{ is closed and } \dim(\ker(A^\dagger)) < \infty\}.$$

□

Remark 6.22. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$.

- (a) The so-called Weyl operators are Fredholm operators on $V_{\mathbb{H}}^R$ with null index. That is, the set of all Weyl operators,

$$\mathcal{W}(V_{\mathbb{H}}^R) = \{A \in \mathcal{F}(V_{\mathbb{H}}^R) \mid \text{ind}(A) = 0\}.$$

- (b) Since, by Remark 6.9 (e), $A \in \mathcal{F}(V_{\mathbb{H}}^R) \Leftrightarrow A^\dagger \in \mathcal{F}(V_{\mathbb{H}}^R)$ and $\text{ind}(A) = -\text{ind}(A^\dagger)$, $A \in \mathcal{W}(V_{\mathbb{H}}^R) \Leftrightarrow A^\dagger \in \mathcal{W}(V_{\mathbb{H}}^R)$.
- (c) By Proposition 6.5 and Lemma 6.15, if F is a finite rank operator, then $\mathbb{I}_{V_{\mathbb{H}}^R} + F \in \mathcal{W}(V_{\mathbb{H}}^R)$.
- (d) By Theorem 6.13, $A, B \in \mathcal{W}(V_{\mathbb{H}}^R) \Rightarrow AB \in \mathcal{W}(V_{\mathbb{H}}^R)$.
- (e) By Theorem 6.16, $A \in \mathcal{W}(V_{\mathbb{H}}^R), K \in \mathcal{B}_0(V_{\mathbb{H}}^R) \Rightarrow A + K \in \mathcal{W}(V_{\mathbb{H}}^R)$.
- (f) By Corollary 6.18, $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ is invertible, and then $A \in \mathcal{W}(V_{\mathbb{H}}^R)$.
- (g) Suppose $\dim(V_{\mathbb{H}}^R) < \infty$, then $\dim(V_{\mathbb{H}}^R) = \dim(\ker(A)) + \dim(\text{ran}(A))$. Also $V_{\mathbb{H}}^R = \overline{\text{ran}(A)} + \text{ran}(A)^\perp$. Hence $\dim(V_{\mathbb{H}}^R) = \dim(\text{ran}(A)) + \dim(\text{ran}(A)^\perp)$. Thus,

$$\begin{aligned} \text{ind}(A) &= \dim(\ker(A)) - \dim(\ker(A^\dagger)) \\ &= \dim(\ker(A)) - \dim(\text{ran}(A)^\perp) \\ &= \dim(\ker(A)) + \dim(\text{ran}(A)) - \dim(V_{\mathbb{H}}^R) = 0. \end{aligned}$$

Therefore, in the finite dimensional space, every operator in $\mathcal{B}(V_{\mathbb{H}}^R)$ is a Fredholm operator with index zero. In this case, $\mathcal{W}(V_{\mathbb{H}}^R) = \mathcal{B}(V_{\mathbb{H}}^R)$.

Remark 6.23. In the complex case, the Fredholm theory can also be extended to unbounded operators using the graph norm. For such a complex treatment, we refer the reader to, for example, Ref. 14. Following the complex theory, we may be able to extend the quaternionic Fredholm theory to quaternionic unbounded operators.

VII. ESSENTIAL S-SPECTRUM

In this section, we define and study the essential S-spectrum in $V_{\mathbb{H}}^R$ in terms of the Fredholm operators. In particular, we give an interesting characterization to the S-spectrum in terms of the Fredholm operators and the Fredholm index (see Proposition 7.18).

Theorem 7.1 (Ref. 13). *Let $V_{\mathbb{H}}^R$ be a right quaternionic Hilbert space equipped with a left scalar multiplication. Then the set $\mathcal{B}(V_{\mathbb{H}}^R)$ equipped with the point-wise sum, with the left and right scalar multiplications defined in Eqs. (3.5) and (3.6), with the composition as product, with the adjunction $A \rightarrow A^\dagger$, as in (3.2), as $*$ -involution, and with the norm defined in (3.1) is a quaternionic two-sided Banach C^* -algebra with unity $\mathbb{I}_{V_{\mathbb{H}}^R}$.*

Remark 7.2. In the above theorem, if the left scalar multiplication is left out on $V_{\mathbb{H}}^R$, then $\mathcal{B}(V_{\mathbb{H}}^R)$ becomes a real Banach C^* -algebra with unity $\mathbb{I}_{V_{\mathbb{H}}^R}$.

Theorem 7.3. *The set of all compact operators, $\mathcal{B}_0(V_{\mathbb{H}}^R)$ is a closed biideal of $\mathcal{B}(V_{\mathbb{H}}^R)$ and is closed under adjunction.*

Proof. Theorem 7.1 gives together with Proposition 4.2 that $\mathcal{B}_0(V_{\mathbb{H}}^R)$ is a closed biideal of $\mathcal{B}(V_{\mathbb{H}}^R)$. Theorem 4.9 is enough to conclude that $\mathcal{B}_0(V_{\mathbb{H}}^R)$ is closed under adjunction. Hence the theorem holds. □

In the quotient space $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$, the coset of $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ is

$$[A] = \{S \in \mathcal{B}(V_{\mathbb{H}}^R) \mid S = A + K \text{ for some } K \in \mathcal{B}_0(V_{\mathbb{H}}^R)\} = A + \mathcal{B}_0(V_{\mathbb{H}}^R).$$

In the quotient space, define the product

$$[A][B] = [AB].$$

Since $\mathcal{B}_0(V_{\mathbb{H}}^R)$ is a closed subspace of $\mathcal{B}(V_{\mathbb{H}}^R)$, with the above product, $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$ is a unital Banach algebra with unit $[\mathbb{I}_{V_{\mathbb{H}}^R}]$. We call this algebra the quaternionic Calkin algebra. Define the natural quotient map

$$\pi : \mathcal{B}(V_{\mathbb{H}}^R) \rightarrow \mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R) \text{ by } \pi(A) = [A] = A + \mathcal{B}_0(V_{\mathbb{H}}^R).$$

Note that $[0] = \mathcal{B}_0(V_{\mathbb{H}}^R)$ and hence

$$\ker(\pi) = \{A \in \mathcal{B}(V_{\mathbb{H}}^R) \mid \pi(A) = [0]\} = \mathcal{B}_0(V_{\mathbb{H}}^R).$$

Since $\mathcal{B}_0(V_{\mathbb{H}}^R)$ is an ideal of $\mathcal{B}(V_{\mathbb{H}}^R)$, for $A, B \in \mathcal{B}(V_{\mathbb{H}}^R)$, we have

- (a) $\pi(A + B) = (A + B) + \mathcal{B}_0(V_{\mathbb{H}}^R) = (A + \mathcal{B}_0(V_{\mathbb{H}}^R)) + (B + \mathcal{B}_0(V_{\mathbb{H}}^R)) = \pi(A) + \pi(B)$.
- (b) $\pi(AB) = AB + \mathcal{B}_0(V_{\mathbb{H}}^R) = (A + \mathcal{B}_0(V_{\mathbb{H}}^R))(B + \mathcal{B}_0(V_{\mathbb{H}}^R)) = \pi(A)\pi(B)$.
- (c) $\pi(\mathbb{I}_{V_{\mathbb{H}}^R}) = [\mathbb{I}_{V_{\mathbb{H}}^R}]$.

Hence π is a unital homomorphism. The norm on $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$ is given by

$$\|[A]\| = \inf_{K \in \mathcal{B}_0(V_{\mathbb{H}}^R)} \|A + K\| \leq \|A\|.$$

Therefore π is a contraction.

Theorem 7.4. *Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$. The following are pairwise equivalent:*

- (a) $A \in \mathcal{F}_l(V_{\mathbb{H}}^R)$.
- (b) There exist $S \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $K \in \mathcal{B}_0(V_{\mathbb{H}}^R)$ such that $SA = \mathbb{I}_{V_{\mathbb{H}}^R} + K$.
- (c) There exist $S \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $K_1, K_2, K_3 \in \mathcal{B}_0(V_{\mathbb{H}}^R)$ such that $(S + K_1)(A + K_2) = \mathbb{I}_{V_{\mathbb{H}}^R} + K_3$.
- (d) $\pi(S)\pi(A) = \pi(\mathbb{I}_{V_{\mathbb{H}}^R})$, the identity in $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$, for some $\pi(S) \in \mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$.
- (e) $\pi(A)$ is left invertible in $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$.

Proof. By the definition of left semi-Fredholm operators, (a) and (b) are equivalent.

(b) \Rightarrow (c): Suppose (b) holds. Since $(S + K_1)(A + K_2) = SA + SK_2 + K_1A + K_1K_2 = \mathbb{I}_{V_{\mathbb{H}}^R} + K + SK_2 + K_1A + K_1K_2 = \mathbb{I}_{V_{\mathbb{H}}^R} + K_3$, by Proposition 4.2, $K_3 = K + SK_2 + K_1A + K_1K_2$ is compact.

(c) \Rightarrow (d): Suppose (c) holds. Then there exist $S_1 \in [S] = \pi(S)$, $A_1 \in [A] = \pi(A)$, and $J \in [\mathbb{I}_{V_{\mathbb{H}}^R}] = \pi(\mathbb{I}_{V_{\mathbb{H}}^R})$ such that $S_1A_1 = J$. Hence $\pi(S)\pi(A) = [S][A] = [SA] = [J] = [\mathbb{I}_{V_{\mathbb{H}}^R}] = \pi(\mathbb{I}_{V_{\mathbb{H}}^R})$. Therefore (d) is established.

(d) \Rightarrow (b): Suppose (d) holds. $B \in \pi(S)\pi(A) = \pi(SA)$ if and only if $B = SA + K_1$ for some $K_1 \in \mathcal{B}_0(V_{\mathbb{H}}^R)$. Also $B \in \pi(\mathbb{I}_{V_{\mathbb{H}}^R}) = [\mathbb{I}_{V_{\mathbb{H}}^R}]$ if and only if $B = \mathbb{I}_{V_{\mathbb{H}}^R} + K_2$ for some $K_2 \in \mathcal{B}_0(V_{\mathbb{H}}^R)$. Therefore, $SA = \mathbb{I}_{V_{\mathbb{H}}^R} + K_2 - K_1 = \mathbb{I}_{V_{\mathbb{H}}^R} + K$, where $K = K_2 - K_1 \in \mathcal{B}_0(V_{\mathbb{H}}^R)$. Thus $A \in \mathcal{F}_l(V_{\mathbb{H}}^R)$.

(a) and (e) are equivalent by the definition of left invertibility. □

Corollary 7.5. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$. $A \in \mathcal{F}_l(V_{\mathbb{H}}^R)$ (or $A \in \mathcal{F}_r(V_{\mathbb{H}}^R)$) if and only if $\pi(A)$ is left (or right) invertible in the Calkin algebra $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$.

Proof. It is straightforward from Theorem 7.4 for $\mathcal{F}_l(V_{\mathbb{H}}^R)$. Also a similar version of Theorem 7.4 holds for $\mathcal{F}_r(V_{\mathbb{H}}^R)$. □

Definition 7.6. The essential S -spectrum (or the Calkin S -spectrum) $\sigma_e^S(A)$ of $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ is the S -spectrum of $\pi(A)$ in the unital Banach algebra $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$. That is,

$$\sigma_e^S(A) = \sigma_S(\pi(A)).$$

Similarly, the left essential S -spectrum $\sigma_{el}^S(A)$ and the right essential S -spectrum $\sigma_{er}^S(A)$ are the left S -spectrum and right S -spectrum of $\pi(A)$, respectively. That is,

$$\sigma_{el}^S(A) = \sigma_l^S(\pi(A)) \quad \text{and} \quad \sigma_{er}^S(A) = \sigma_r^S(\pi(A))$$

in $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$.

Clearly, by definition, $\sigma_e^S(A) = \sigma_{el}^S(A) \cup \sigma_{er}^S(A)$ and $\sigma_e^S(A)$ is a compact subset of \mathbb{H} .

Proposition 7.7. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, and then

$$\sigma_{el}^S(A) = \{q \in \mathbb{H} \mid R_q(A) \in \mathcal{B}(V_{\mathbb{H}}^R) \setminus \mathcal{F}_l(V_{\mathbb{H}}^R)\}, \tag{7.1}$$

$$\sigma_{er}^S(A) = \{q \in \mathbb{H} \mid R_q(A) \in \mathcal{B}(V_{\mathbb{H}}^R) \setminus \mathcal{F}_r(V_{\mathbb{H}}^R)\}. \tag{7.2}$$

Proof. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$. Then by Corollary 7.5, $R_q(A) \notin \mathcal{F}_l(V_{\mathbb{H}}^R)$ if and only if $\pi(R_q(A))$ is not left invertible in $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$, which means, by the definition of left S -spectrum, $q \in \sigma_l^S(\pi(A))$. That is, $q \in \sigma_{el}^S(A)$ if and only if $R_q(A) \notin \mathcal{F}_l(V_{\mathbb{H}}^R)$. Hence we have Eq. (7.1). A similar argument proves Eq. (7.2). □

Corollary 7.8 (Atkinson theorem). Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, and then

$$\sigma_e^S(A) = \{q \in \mathbb{H} \mid R_q(A) \in \mathcal{B}(V_{\mathbb{H}}^R) \setminus \mathcal{F}(V_{\mathbb{H}}^R)\}. \tag{7.3}$$

Proof. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$. Since $\sigma_e^S(A) = \sigma_{el}^S(A) \cup \sigma_{er}^S(A)$ and $\mathcal{F}(V_{\mathbb{H}}^R) = \mathcal{F}_l(V_{\mathbb{H}}^R) \cap \mathcal{F}_r(V_{\mathbb{H}}^R)$, it is straightforward from Proposition 7.7. □

Corollary 7.9 (Atkinson theorem). Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$. A is Fredholm if and only if $\pi(A)$ is invertible in the Calkin algebra $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$.

Proof. $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ is Fredholm if and only if A is left and right semi-Fredholm. Furthermore, $\pi(A)$ is invertible in $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$ if and only if $\pi(A)$ is both left and right invertible in $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$. Therefore, from Corollary 7.5, we have the desired result. \square

Proposition 7.10. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, and then $\sigma_{el}^S(A)$ and $\sigma_{er}^S(A)$ are closed subsets of \mathbb{H} .

Proof. In the unital algebra $\mathcal{B}(V_{\mathbb{H}}^R)$, by Theorems 5.4 and 5.5, $\sigma_l^S(A) = \sigma_{ap}^S(A)$ and $\sigma_r^S(A) = \sigma_{ap}^S(A^\dagger)$ are closed subsets of \mathbb{H} . By the same argument, in the Calkin algebra $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$, $\sigma_{el}^S(A) = \sigma_l^S(\pi(A))$ and $\sigma_{er}^S(A) = \sigma_r^S(\pi(A))$ are closed subsets of \mathbb{H} . \square

Proposition 7.11. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, and then

$$\sigma_{el}^S(A) = \{q \in \mathbb{H} \mid \text{ran}(R_q(A)) \text{ is not closed or } \dim(\ker(R_q(A))) = \infty\}, \tag{7.4}$$

$$\sigma_{er}^S(A) = \{q \in \mathbb{H} \mid \text{ran}(R_q(A)) \text{ is not closed or } \dim(\ker(R_q(A^\dagger))) = \infty\}. \tag{7.5}$$

Proof. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$. By Proposition 7.7, $q \in \sigma_{el}^S(A)$ if and only if $R_q(A)$ is not left semi-Fredholm. By Theorem 6.21, $R_q(A)$ is not left semi-Fredholm if and only if $\text{ran}(R_q(A))$ is not closed or $\dim(\ker(R_q(A))) = \infty$. Hence we have Eq. (7.4). In the same way, Eq. (7.5) can be obtained. \square

Corollary 7.12. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, and then

$$\sigma_{el}^S(A) = \sigma_{er}^S(A^\dagger), \quad \sigma_{er}^S(A) = \sigma_{el}^S(A^\dagger), \quad \text{and hence} \quad \sigma_e^S(A) = \sigma_e^S(A^\dagger).$$

Proof. Since $R_q(A) = R_{\bar{q}}(A)$, by Proposition 3.8, $\text{ran}(R_q(A))$ is closed if and only if $\text{ran}(R_{\bar{q}}(A^\dagger))$ is closed, and $\sigma_e^S(A) = \sigma_{er}^S(A) \cup \sigma_{el}^S(A)$, Proposition 7.11 concludes the results. \square

Remark 7.13. If $\dim(V_{\mathbb{H}}^R) < \infty$, then all operators in $\mathcal{B}(V_{\mathbb{H}}^R)$ are finite rank operators and hence compact. Therefore, the Calkin algebra $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$ is null. Hence, in this case, $\sigma_e^S(A) = \emptyset$.

Proposition 7.14. For $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, $\sigma_e^S(A) \neq \emptyset$ if and only if $\dim(V_{\mathbb{H}}^R) = \infty$.

Proof. (\Leftarrow) Suppose $\dim(V_{\mathbb{H}}^R) = \infty$. $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$ is a unital Banach algebra. By Proposition 3.16, the S -spectrum on a unital Banach algebra is not empty. Therefore $\sigma_e^S(A) = \sigma_S(\pi(A)) \neq \emptyset$.

(\Rightarrow) By Remark 7.13, $\dim(V_{\mathbb{H}}^R) < \infty$ implies $\sigma_e^S(A) = \emptyset$. That is, $\sigma_e^S(A) \neq \emptyset$ implies $\dim(V_{\mathbb{H}}^R) = \infty$. \square

Proposition 7.15. For every $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $K \in \mathcal{B}_0(V_{\mathbb{H}}^R)$, we have $\sigma_e^S(A + K) = \sigma_e^S(A)$. In the same way, $\sigma_{el}^S(A + K) = \sigma_{el}^S(A)$ and $\sigma_{er}^S(A + K) = \sigma_{er}^S(A)$.

Proof. For every $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $K \in \mathcal{B}_0(V_{\mathbb{H}}^R)$, we have $\pi(A + K) = \pi(A)$ in $\mathcal{B}(V_{\mathbb{H}}^R)/\mathcal{B}_0(V_{\mathbb{H}}^R)$. Therefore by the definition, we get

$$\sigma_e^S(A + K) = \sigma_S(\pi(A + K)) = \sigma_S(\pi(A)) = \sigma_e^S(A).$$

The other two equalities follow in the same way. \square

Proposition 7.16. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$. Then,

$$\sigma_{el}^S(A) \subseteq \sigma_l^S(A) \quad \text{and} \quad \sigma_{er}^S(A) \subseteq \sigma_r^S(A),$$

and hence $\sigma_e^S(A) \subseteq \sigma_S(A)$.

Proof. It is straightforward from Propositions 3.18 and 7.10. □

Definition 7.17. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $k \in \mathbb{Z} \setminus \{0\}$. Define

$$\sigma_k^S(A) = \{q \in \mathbb{H} \mid R_q(A) \in \mathcal{F}(V_{\mathbb{H}}^R) \quad \text{and} \quad \text{ind}(R_q(A)) = k\}.$$

Also

$$\sigma_0^S = \{q \in \sigma_S(A) \mid R_q(A) \in \mathcal{W}(V_{\mathbb{H}}^R)\}.$$

Proposition 7.18. Let $A \in \mathcal{B}(V_{\mathbb{H}}^R)$, and then $\sigma_S(A) = \sigma_e^S(A) \cup \bigcup_{k \in \mathbb{Z}} \sigma_k^S(A)$.

Proof. Clearly the family $\{\sigma_k^S(A)\}_{k \in \mathbb{Z}}$ is pair-wise disjoint. Let $0 \neq k \in \mathbb{Z}$ and $q \in \sigma_k^S(A)$. If $k > 0$, then $0 < \text{ind}(R_q(A)) < \infty$, and hence $0 < \dim(\ker(R_q(A))) - \dim(\ker(R_q(A^\dagger))) < \infty$. Thus, $\ker(R_q(A)) \neq \{0\}$ and therefore $q \in \sigma_S(A)$. If $k < 0$, then $\dim(\ker(R_q(A))) < \dim(\ker(R_q(A^\dagger)))$. Hence $\ker(R_q(A^\dagger)) \neq \{0\}$ and therefore $R_q(A^\dagger)$ is not invertible. Thus, by Proposition 3.9, $R_q(A)$ is not invertible. That is, $q \in \sigma_S(A)$. Altogether we get

$$\bigcup_{k \in \mathbb{Z}} \sigma_k^S(A) = \{q \in \sigma_S(A) \mid R_q(A) \in \mathcal{F}(V_{\mathbb{H}}^R)\}.$$

Also, by Proposition 7.16, we have $\sigma_e^S(A) \subseteq \sigma_S(A)$ and by Corollary 7.8, $\sigma_e^S(A) = \{q \in \sigma_S(A) \mid R_q(A) \notin \mathcal{F}(V_{\mathbb{H}}^R)\}$. Therefore, $\sigma_S(A) = \sigma_e^S(A) \cup \bigcup_{k \in \mathbb{Z}} \sigma_k^S(A)$, clearly with $\sigma_e^S(A) \cap \bigcup_{k \in \mathbb{Z}} \sigma_k^S(A) = \emptyset$. □

VIII. CONCLUSION

We have studied the approximate S-point spectrum, Fredholm operators, and essential S-spectrum of a bounded right linear operator in a right quaternionic Hilbert space $V_{\mathbb{H}}^R$. The left multiplication defined on $V_{\mathbb{H}}^R$ does not play a role in the approximate S-point spectrum and in the Fredholm theory. However, in getting a unital Calkin algebra, it has appeared and hence played a role in the S-essential spectrum. Also an interesting characterization to the S-spectrum is given in terms of the Fredholm operators and its index.

In the complex case, the exact energy spectrum is obtainable and known only for a handful of Hamiltonians, and for the rest only approximate spectrum is computed numerically. In these approximate theories, the Fredholm operators and the essential spectra are used frequently. Furthermore, these theories are successfully used in transport operator, integral operator, and differential operator problems.^{3,12,15,16,18,8} In the same manner, the theories developed in this note may be useful in obtaining the approximate S-point spectrum or essential S-spectrum for quaternionic Hamiltonians and their small norm perturbations. For quaternionic Hamiltonians and potentials, see Refs. 1 and 9 and the references therein.

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