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REGULAR ARTICLE

Optimal stochastic restricted logistic estimator

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Abstract
It is well known that the use of prior information in the logistic regression improves the estimates of regression coefficients when multicollinearity presents. This prior information may be in the form of exact or stochastic linear restrictions. In this article, in the presence of stochastic linear restrictions, we propose a new efficient estimator, named Stochastic restricted optimal logistic estimator for the parameters in the logistic regression models when the multicollinearity presents. Further, conditions for the superiority of the new optimal estimator over some existing estimators are derived with respect to the mean square error matrix sense. Moreover, a Monte Carlo simulation study and a real data example are provided to illustrate the performance of the proposed optimal estimator in the scalar mean square error sense.

Keywords Logistic regression · Multicollinearity · Optimal estimator · Mean square error · Scalar mean square error

1 Introduction

Logistic regression was developed by Cox (1958), and is used to explain the relationship between one dependent binary variable and one or more nominal, ordinal, interval or ratio-level independent variables. The applications of logistic regression model are in various fields, including machine learning, medical fields, and social sciences. The logistic regression model is defined as

$$y_i = \pi_i + \varepsilon_i, \quad i = 1, \ldots, n, \tag{1.1}$$

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where the response y_i follows Binary distribution with parameter π_i as

$$\pi_i = \frac{\exp(x_i'\beta)}{1+\exp(x_i'\beta)}, \tag{1.2}$$

with x_i is the ith row of X, which is an $n \times p$ data matrix with p explanatory variables and β is a $p \times 1$ vector of coefficients, ε_i are independent with mean zero and variance $\pi_i(1-\pi_i)$ of the response y_i. The maximum likelihood estimation method is the commonly used estimation technique to estimate the regression parameter β , and it depends on the likelihood function of the logistic regression model (1.1),

$$L(\beta) = \prod_{i=1}^{n} \pi_i^{y_i}(1-\pi_i)^{(1-y_i)}. \tag{1.3}$$

Then the likelihood equation to estimate the parameters becomes

$$\frac{\partial \ell(\beta)}{\partial \beta} = \sum_{i=1}^{n}(y_i - \pi_i)x_i = 0, \tag{1.4}$$

where

$$\ell(\beta) = \sum_{i=1}^{n} y_i \ln(\pi_i) + \sum_{i=1}^{n}(1-y_i)\ln(1-\pi_i), \tag{1.5}$$

is the log-likelihood function of the logistic regression model. Now, the maximum likelihood estimator (MLE) of β can be obtained by solving Eq. (1.4). Since Eq. (1.4) is nonlinear in parameter β, the Iterative weighted least square algorithm can be used to estimate the parameter β, and the corresponding maximum likelihood estimator takes the form

$$\hat{\beta}_{MLE} = C^{-1}X'\hat{W}Z, \tag{1.6}$$

where $C = X'\hat{W}X$; Z is the column vector with i^{th} element equals $logit(\hat{\pi}_i) + \frac{y_i - \hat{\pi}_i}{\hat{\pi}_i(1-\hat{\pi}_i)}$ and $\hat{W} = diag[\hat{\pi}_i(1-\hat{\pi}_i)]$. Note that $\hat{\beta}_{MLE}$ is asymptotically unbiased for β and its asymptotic covariance matrix is given by

$$Cov(\hat{\beta}_{MLE}) = \{X'\hat{W}X\}^{-1}. \tag{1.7}$$

Then the mean square error matrix (MSEM) of $\hat{\beta}_{MLE}$ is given by

$$\begin{aligned} MSEM[\hat{\beta}_{MLE}] &= Cov[\hat{\beta}_{MLE}] + B[\hat{\beta}_{MLE}]B'[\hat{\beta}_{MLE}] \\ &= \{X'\hat{W}X\}^{-1} \\ &= C^{-1}. \end{aligned} \tag{1.8}$$

Since the explanatory variables are multicollinear in many practical situations, the maximum likelihood estimation method produces inefficient estimates. Several studies have been done to propose alternative estimators to MLE to overcome the multicollinearity problem. These estimators can be categorized mainly to two cases: (i) estimators based on the sample information (1.1), and (ii) estimators based on both the sample and the prior information. The prior information can be in the form of exact linear restrictions or stochastic linear restrictions. In practice, the exact or stochastic restrictions can be chosen based on the past experience related to the study or on the similar kind of experiments conducted in the past.

According to the literature, the estimators based on the sample information are; Logistic Ridge Estimator (LRE) (Schaefer et al. 1984), Principal Component Logistic Estimator (Aguilera et al. 2006), Modified Logistic Ridge Estimator (Nja et al. 2013), Logistic Liu Estimator (LLE) (Mansson et al. 2012), Liu-Type Logistic Estimator (Inan and Erdogan 2013), Almost Unbiased Ridge Logistic Estimator (AURLE) (Wu and Asar 2016), and Almost Unbiased Liu Logistic Estimator (AULLE) (Xinfeng 2015).

The estimators based on the sample and prior information in the form of exact linear restrictions are; Restricted Maximum Likelihood Estimator (Duffy and Santner 1989), Restricted Logistic Liu Estimator (Şiray et al. 2015), Modified Restricted Liu Estimator (Wu 2016), Restricted Logistic Ridge Estimator (Asar et al. 2017a), Restricted Liu-Type Logistic Estimator (Asar et al. 2017b), Restricted Almost Unbiased Ridge logistic Estimator (Varathan and Wijekoon 2016a). The estimators based on the sample and prior information in the form of stochastic linear restrictions are; Stochastic Restricted Maximum Likelihood Estimator (SRMLE) (Nagarajah and Wijekoon 2015), Stochastic Restricted Ridge Maximum Likelihood Estimator (SRRMLE) (Varathan and Wijekoon 2016b), Stochastic Restricted Liu Maximum Likelihood Estimator (SRLMLE) (Varathan and Wijekoon 2016c), and Stochastic Restricted Liu-type logistic estimator (SRLTLE) (Varathan and Wijekoon 2018b), Stochastic Restricted Almost Unbiased Ridge Logistic Estimator (SRAURLE) (Varathan and Wijekoon 2017) and Stochastic Restricted Almost Unbiased Liu Logistic Estimator (SRAULLE) (Varathan and Wijekoon 2018c).

Moreover, Varathan and Wijekoon (2018a) have introduced an Optimal Generalized Logistic Estimator (OGLE) for the logistic regression based on the sample information. Further, in their study, the performance of the proposed estimator was compared with some existing logistic estimators such as MLE, LRE, LLE, AURLE, and AULLE in the scalar mean square error criterion.

In the present study, we introduce a new estimator for the logistic regression based on the sample information and the prior information in the form of stochastic linear restrictions. This new estimator is named as stochastic restricted optimal logistic estimator (SROLE), and the conditions for superiority of the new estimator over the existing estimators OGLE, SRMLE, SRRMLE, SRLMLE, SRAULLE, SRAURLE, and SRLTLE are derived by means of the mean square error matrix criterion. Further, by conducting a simulation study, and analyzing a numerical example, we compare the performance of the proposed estimator SROLE with the other existing estimators in the scalar mean square error sense.

The rest of the article is organized as follows. The construction of the proposed estimator is given in Sect. 2. In Sect. 3, the conditions for superiority of the proposed estimator SROLE over the other existing estimators are derived with respect to mean square error matrix criterion. In Sect. 4, the results of a Monte Carlo simulation study is presented to understand the performance of the proposed estimator over the other estimators in the scalar mean square error (SMSE) sense. In Sect. 5, results related to a numerical example is shown to illustrate the theoretical findings. Finally, some conclusive remarks are given in Sect. 6.

2 Construction of proposed optimal estimator

Consider the following stochastic linear restriction in addition to the sample logistic regression model (1.1).

$$h = H\beta + \upsilon; \quad E(\upsilon) = \mathbf{0}, \quad Cov(\upsilon) = \Psi. \tag{2.1}$$

where h is an $(q \times 1)$ stochastic known vector, H is a $(q \times p)$ of full rank $q \leq p$ known elements and υ is an $(q \times 1)$ random vector of disturbances with mean $\mathbf{0}$ and dispersion matrix Ψ, which is assumed to be known $(q \times q)$ positive definite matrix. Further, it is assumed that υ is statistically independent of ε, i.e., $E(\varepsilon\upsilon') = 0$.

2.1 Existing estimators

Nagarajah and Wijekoon (2015), introduced the stochastic restricted maximum likelihood estimator (SRMLE)

$$\hat{\beta}_{SRMLE} = \hat{\beta}_{MLE} + C^{-1}H'(\Psi + HC^{-1}H')^{-1}(h - H\hat{\beta}_{MLE}), \tag{2.2}$$

in the presence of the stochastic linear restrictions defined in (2.1) with the logistic regression model (1.1), to reduce the effect of multicollinearity problem. The asymptotic properties of SRMLE are

$$E(\hat{\beta}_{SRMLE}) = \beta, \tag{2.3}$$

and

$$\begin{aligned} Cov(\hat{\beta}_{SRMLE}) &= C^{-1} - C^{-1}H'(\Psi + HC^{-1}H')^{-1}HC^{-1} \\ &= (C + H'\Psi^{-1}H)^{-1} \\ &= R. \end{aligned} \tag{2.4}$$

Consequently, the mean square error matrix is obtained as

$$\begin{aligned} MSEM(\hat{\beta}_{SRMLE}) &= (C + H'\Psi^{-1}H)^{-1} \\ &= R. \end{aligned} \tag{2.5}$$

Based on the SRMLE, the following estimators have been proposed in the literature.

1. Varathan and Wijekoon (2016b);

$$\hat{\beta}_{SRRMLE} = Z_k \hat{\beta}_{SRMLE}, \tag{2.6}$$

where $Z_k = (I + kC^{-1})^{-1},\ k \geq 0.$

2. Varathan and Wijekoon (2016c);

$$\hat{\beta}_{SRLMLE} = Z_d \hat{\beta}_{SRMLE}, \tag{2.7}$$

where $Z_d = (C + I)^{-1}(C + dI),\ 0 \leq d \leq 1.$

3. Varathan and Wijekoon (2017);

$$\hat{\beta}_{SRAURLE} = W_k \hat{\beta}_{SRMLE}, \tag{2.8}$$

where $W_k = [I - k^2(C + kI)^{-2}],\ k \geq 0.$

4. Varathan and Wijekoon (2018c);

$$\hat{\beta}_{SRAULLE} = W_d \hat{\beta}_{SRMLE}, \tag{2.9}$$

where $W_d = [I - (1 - d)^2(C + I)^{-2}],\ 0 \leq d \leq 1.$

5. Varathan and Wijekoon (2018b);

$$\hat{\beta}_{SRLTLE} = Z_{k,d} \hat{\beta}_{SRMLE}, \tag{2.10}$$

where $Z_{k,d} = (C + kI)^{-1}(C - dI),\ k \geq 0\ ;0 \leq d \leq 1.$

Since the estimators defined in (2.6)–(2.10) have a common structure, one can write the general form of the estimators SRRMLE, SRLMLE, SRAURLE, SRAULLE, and SRLTLE as

$$\hat{\beta}_{SRGLE} = F \hat{\beta}_{SRMLE}, \tag{2.11}$$

where F is a non negative definite matrix, and note that

$$\hat{\beta}_{SRGLE} = \begin{cases} \hat{\beta}_{SRMLE} & \text{if } F = I; \\ \hat{\beta}_{SRRMLE} & \text{if } F = Z_k; \\ \hat{\beta}_{SRLMLE} & \text{if } F = Z_d; \\ \hat{\beta}_{SRAURLE} & \text{if } F = W_k; \\ \hat{\beta}_{SRAULLE} & \text{if } F = W_d; \\ \hat{\beta}_{SRLTLE} & \text{if } F = Z_{k,d}. \end{cases} \tag{2.12}$$

We name the above estimator $\hat{\beta}_{SRGLE}$ as the stochastic restricted generalized logistic estimator (SRGLE), and the bias vector, dispersion matrix, and mean square error matrix of $\hat{\beta}_{SRGLE}$ can be obtained as

$$\begin{aligned} Bias(\hat{\beta}_{SRGLE}) &= E[\hat{\beta}_{SRGLE}] - \beta \\ &= [F - I]\beta, \end{aligned} \tag{2.13}$$

$$\begin{aligned} D(\hat{\beta}_{SRGLE}) &= Cov(\hat{\beta}_{SRGLE}) \\ &= Cov(F\hat{\beta}_{SRMLE}) \\ &= FRF', \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} MSEM(\hat{\beta}_{SRGLE}) &= D(\hat{\beta}_{SRGLE}) + Bias(\hat{\beta}_{SRGLE})Bias(\hat{\beta}_{SRGLE})' \\ &= FRF' + (F - I)\beta\beta'(F - I)', \end{aligned} \tag{2.15}$$

where $R = (C + H'\Psi^{-1}H)^{-1}$, respectively. Consequently, the scalar mean square error can be obtained as

$$\begin{aligned} SMSE(\hat{\beta}_{SRGLE}) &= tr[FRF'] + \beta'(F - I)'(F - I)\beta \\ &= tr[FRF'] + \beta'(I - F^{-1})'F'F(I - F^{-1})\beta. \end{aligned} \tag{2.16}$$

Note that, using the above general form, we can easily derive the stochastic properties of the estimators SRMLE, SRRMLE, SRLMLE, SRAURLE, SRAULLE, and SRLTLE or any other estimator in the form of SRGLE by changing the non negative definite matrix F.

2.2 Proposed estimator

Note that the matrix F defined in the estimator SRGLE takes different choices depending on different estimators. Therefore, it is better to find an optimal form for F. To achieve this, we minimize the scalar mean square error of SRGLE with respect to F. Consider the derivative of Eq. (2.16) with respect to F

$$\begin{aligned} \frac{\partial\{SMSE(\hat{\beta}_{SRGLE})\}}{\partial F} &= \frac{\partial\{tr(FRF')\}}{\partial F} + \frac{\partial\beta'(I - F^{-1})'F'F(I - F^{-1})\beta}{\partial F} \\ &= \frac{\partial\{tr(FRF')\}}{\partial F} + \frac{\partial\{\beta'F'F\beta - 2\beta'F\beta + \beta'\beta\}}{\partial F} \\ &= \frac{\partial\{tr(FRF')\}}{\partial F} + \frac{\partial\{\beta'F'F\beta\}}{\partial F} - 2\frac{\partial\{\beta'F\beta\}}{\partial F}. \end{aligned} \tag{2.17}$$

To simplify the above equation further, we consider the following lemmas (see Rao and Toutenburg 1995, p. 385, 386).

Lemma 1 *Let A and B be any two matrices with proper order, then*

$$\frac{\partial tr(ABA')}{\partial A} = A(B + B').$$

Lemma 2 *If a is a vector of order $n \times 1$, b is another vector of order $m \times 1$, and M is an $n \times m$ matrix, then*

$$\frac{\partial a' M b}{\partial M} = ab'.$$

Lemma 3 *Let a be a $n \times 1$ vector, N a symmetric $t \times t$ matrix, and M a $t \times n$ matrix, then*

$$\frac{\partial a' M' N M a}{\partial M} = 2NMaa'.$$

By applying Lemmas 1, 2, and 3 in (2.17), we obtain

$$\begin{aligned}\frac{\partial \{SMSE(\hat{\beta}_{SRGLE})\}}{\partial F} &= 2FR + 2F\beta\beta' - 2\beta\beta' \\ &= 2F(R + \beta\beta') - 2\beta\beta'. \end{aligned} \tag{2.18}$$

Note that the matrix $R + \beta\beta'$ is positive definite (see Rao and Toutenburg 1995, p. 366), and hence, non-singular.
By equating (2.18) to a null-matrix, we shall obtain an optimal choice of F as,

$$\tilde{F}_{Opt} = \beta\beta'(R + \beta\beta')^{-1}. \tag{2.19}$$

Now we are ready to propose a new estimator named stochastic restricted optimal logistic estimator (SROLE) as below:

$$\hat{\beta}_{SROLE} = \tilde{F}_{Opt}\hat{\beta}_{SRMLE}. \tag{2.20}$$

Since the above optimal estimator SROLE contains an unknown parameter β within the term $\tilde{F}_{Opt}$ [see (2.19)], it is necessary to identify a suitable vector having known values for β. As such, following Newhouse and Oman (1971), in place of β, we use the normalized eigen vector corresponding to the largest eigen value of the matrix $X'\hat{W}X$ which satisfies the constraint $\beta'\beta = 1$. However, a researcher cannot obtain $X'\hat{W}X$ since $\hat{W}$ is a function of β. Therefore, to construct $\hat{W}$, we choose initial values of the parameters $\beta = (\beta_1, \ldots, \beta_p)'$ according to Mansson and Shukur (2011) and Şiray et al. (2015) such that $\beta_1 = \cdots = \beta_p$ and $\sum_{j=1}^{p} \beta_j^2 = 1$. Then, by substituting the normalized eigen vector corresponding to the largest eigen value of the matrix $X'\hat{W}X$ in place of β, $\tilde{F}_{Opt}$ can be estimated.

2.3 Asymptotic properties of optimal estimator

In this subsection, we summarize the asymptotic properties of the proposed optimal estimator SROLE. Since the present optimal estimator (SROLE) is in the similar form of the estimator SRGLE, by replacing F by $\tilde{F}_{Opt}$ in the Eqs. (2.13)–(2.16), one can easily obtain the following properties.

Bias vector

$$Bias(\hat{\beta}_{SROLE}) = [\tilde{F}_{Opt} - I]\beta, \tag{2.21}$$

Dispersion matrix

$$D(\hat{\beta}_{SROLE}) = \tilde{F}_{Opt} R \tilde{F}'_{Opt}, \tag{2.22}$$

Mean square error matrix

$$MSEM(\hat{\beta}_{SROLE}) = \tilde{F}_{Opt} R \tilde{F}'_{Opt} + (\tilde{F}_{Opt} - I)\beta\beta'(\tilde{F}_{Opt} - I)', \tag{2.23}$$

and Scalar mean square error

$$SMSE(\hat{\beta}_{SROLE}) = tr[\tilde{F}_{Opt} R \tilde{F}'_{Opt}] + \beta'(\tilde{F}_{Opt} - I)'(\tilde{F}_{Opt} - I)\beta. \tag{2.24}$$

Note that when prior information is not available, $R = (C + H'\Psi^{-1}H)^{-1}$ is simply equal to C^{-1}. Then $\tilde{F}_{Opt} = \beta\beta'(R + \beta\beta')^{-1}$ in (2.19) becomes $\tilde{J}_{(i)} = \beta\beta'(C^{-1} + \beta\beta')^{-1}$. Also, only when sample information exists, SRMLE is replaced with MLE. Then the proposed estimator represents the optimal generalized logistic estimator (OGLE) (Varathan and Wijekoon 2018a).

$$\hat{\beta}_{OGLE} = \tilde{J}_{(i)}\hat{\beta}_{MLE}, \tag{2.25}$$

which is a generalized estimator to represent Logistic Ridge Estimator (LRE) (Schaefer et al. 1984), Logistic Liu Estimator (LLE) (Mansson et al. 2012), Almost Unbiased Ridge Logistic Estimator (AURLE) (Wu and Asar 2016), Almost Unbiased Liu Logistic Estimator (AULLE) (Xinfeng 2015), and other logistic estimators of same type based only on sample information.
The properties of optimal generalized logistic estimator (OGLE) are

$$\begin{aligned} Bias(\hat{\beta}_{OGLE}) &= E[\hat{\beta}_{OGLE}] - \beta \\ &= (\tilde{J}_{(i)} - I)\beta, \end{aligned} \tag{2.26}$$

$$D(\hat{\beta}_{OGLE}) = \tilde{J}_{(i)} C^{-1} \tilde{J}'_{(i)}, \tag{2.27}$$

$$MSEM(\hat{\beta}_{OGLE}) = \tilde{J}_{(i)} C^{-1} \tilde{J}'_{(i)} + (\tilde{J}_{(i)} - I)\beta\beta'(\tilde{J}_{(i)} - I)', \tag{2.28}$$

and

$$SMSE(\hat{\beta}_{OGLE}) = tr(\tilde{J}_{(i)} C^{-1} \tilde{J}'_{(i)}) + \beta'(I - \tilde{J}^{-1}_{(i)})'\tilde{J}'_{(i)}\tilde{J}_{(i)}(I - \tilde{J}^{-1}_{(i)})\beta. \tag{2.29}$$

3 Mean square error matrix comparison of estimators

In this section, we examine the performance of the proposed optimal estimator SROLE with the existing estimators SRGLE and OGLE, by means of mean square error matrix criterion.

MSEM criteria
For two given estimators $\hat{\beta}_1$ and $\hat{\beta}_2$, the estimator $\hat{\beta}_2$ is said to be superior to $\hat{\beta}_1$, under the MSEM criterion, if and only if

$$M(\hat{\beta}_1, \hat{\beta}_2) = MSEM(\hat{\beta}_1, \beta) - MSEM(\hat{\beta}_2, \beta) \geq 0. \tag{3.1}$$

To find the superiority condition of the proposed estimator SROLE with the SRGLE, we use following lemmas.

Lemma 4 (Rao et al. 2008) *Let the two $n \times n$ matrices $M > 0$,$N \geq 0$, then $M > N$ if and only if $\lambda_{\max}(NM^{-1}) < 1$.*

Lemma 5 (Trenkler and Toutenburg 1990) *Let $\tilde{\beta}_j = A_j y$, $j = 1, 2$ be two competing homogeneous linear estimators of β. Suppose that $D = Cov(\tilde{\beta}_1) - Cov(\tilde{\beta}_2) > 0$, where $Cov(\tilde{\beta}_j)$, $j = 1, 2$ denotes the covariance matrix of $\tilde{\beta}_j$. Then $\Delta(\tilde{\beta}_1, \tilde{\beta}_2) = MSEM(\tilde{\beta}_1) - MSEM(\tilde{\beta}_2) \geq 0$ if and only if $d_2'(D + d_1'd_1)^{-1}d_2 \leq 1$, where $MSEM(\tilde{\beta}_j)$, d_j; $j = 1, 2$ denote the Mean Square Error Matrix and bias vector of $\tilde{\beta}_j$, respectively.*

3.1 SROLE versus SRGLE

Consider

$$\begin{aligned} & MSEM(\hat{\beta}_{SRGLE}) - MSEM(\hat{\beta}_{SROLE}) \\ & = \{FRF' + (F - I)\beta\beta'(F - I)'\} \\ & \quad - \{\tilde{F}_{Opt}R\tilde{F}'_{Opt} + (\tilde{F}_{Opt} - I)\beta\beta'(\tilde{F}_{Opt} - I)'\} \\ & = \{FRF' - \tilde{F}_{Opt}R\tilde{F}'_{Opt}\} \\ & \quad + \{(F - I)\beta\beta'(F - I)' - (\tilde{F}_{Opt} - I)\beta\beta'(\tilde{F}_{Opt} - I)'\}. \end{aligned} \tag{3.2}$$

Now consider

$$\begin{aligned} D(\hat{\beta}_{SRGLE}) - D(\hat{\beta}_{SROLE}) &= FRF' - \tilde{F}_{Opt}R\tilde{F}'_{Opt} \\ &= D^*. \end{aligned} \tag{3.3}$$

Note that, since R is positive definite matrix, FRF' and $\tilde{F}_{Opt}R\tilde{F}'_{Opt}$ are positive definite matrices (see Rao and Toutenburg 1995, p. 366). Consequently, by Lemma 4, if $\lambda_{\max}(\tilde{F}_{Opt}R\tilde{F}'_{Opt}(FRF')^{-1}) < 1$ then D^* is a positive definite matrix, where $\lambda_{\max}[\tilde{F}_{Opt}R\tilde{F}'_{Opt}(FRF')^{-1}] < 1$ is the largest eigen value of $\tilde{F}_{Opt}R\tilde{F}'_{Opt}(FRF')^{-1}$.

Further, by Lemma 5, $MSE(\hat{\beta}_{SRGLE}) - MSE(\hat{\beta}_{SROLE})$ is non-negative definite if $\delta'_{Opt}[D^* + \delta'_G\delta_G]^{-1}\delta_{Opt} \leq 1$, where $\delta_{Opt} = (\tilde{F}_{Opt} - I)\beta$ and $\delta_G = (F - I)\beta$. Based on the above arguments, we state the following theorem.

Theorem 1 *When $\lambda_{\max}[\tilde{F}_{Opt}R\tilde{F}'_{Opt}(FRF')^{-1}] < 1$, the estimator SROLE is superior than SRGLE if and only if $\delta'_{Opt}[D^* + \delta'_G\delta_G]^{-1}\delta_{Opt} \leq 1$.*

Note that the above Theorem 1 states the necessary and sufficient condition for superiority of the proposed estimator (SROLE) over the general existing estimator (SRGLE). By replacing F by a suitable matrix, one may obtain the following conditions for superiority of SROLE over the existing estimators SRMLE, SRRMLE, SRLMLE, SRAURLE, SRAULLE, and SRLTLE with respect to mean square error.

(i) $F = I$: SROLE is superior than SRMLE if and only if $\lambda_{\max}[\tilde{F}_{Opt}R\tilde{F}'_{Opt}R^{-1}] < 1$ and $\delta'_{Opt}[R - \tilde{F}_{Opt}R\tilde{F}'_{Opt}]^{-1}\delta_{Opt} \leq 1$.

(ii) $F = Z_k$: SROLE is superior than SRRMLE if and only if $\lambda_{\max}[\tilde{F}_{Opt}R\tilde{F}'_{Opt}(Z_kRZ'_k)^{-1}] < 1$ and $\delta'_{Opt}[Z_kRZ'_k - \tilde{F}_{Opt}R\tilde{F}'_{Opt} + \beta'(Z_k - I)'(Z_k - I)\beta]^{-1}\delta_{Opt} \leq 1$.

(iii) $F = Z_d$: SROLE is superior than SRLMLE if and only if $\lambda_{\max}[\tilde{F}_{Opt}R\tilde{F}'_{Opt}(Z_dRZ'_d)^{-1}] < 1$ and $\delta'_{Opt}[Z_dRZ'_d - \tilde{F}_{Opt}R\tilde{F}'_{Opt} + \beta'(Z_d - I)'(Z_d - I)\beta]^{-1}\delta_{Opt} \leq 1$.

(iv) $F = W_k$: SROLE is superior than SRAURLE if and only if $\lambda_{\max}[\tilde{F}_{Opt}R\tilde{F}'_{Opt}(W_kRW'_k)^{-1}] < 1$ and $\delta'_{Opt}[W_kRW'_k - \tilde{F}_{Opt}R\tilde{F}'_{Opt} + \beta'(W_k - I)'(W_k - I)\beta]^{-1}\delta_{Opt} \leq 1$.

(v) $F = W_d$: SROLE is superior than SRAULLE if and only if $\lambda_{\max}[\tilde{F}_{Opt}R\tilde{F}'_{Opt}(W_dRW'_d)^{-1}] < 1$ and $\delta'_{Opt}[W_dRW'_d - \tilde{F}_{Opt}R\tilde{F}'_{Opt} + \beta'(W_d - I)'(W_d - I)\beta]^{-1}\delta_{Opt} \leq 1$.

(vi) $F = Z_{k,d}$: SROLE is superior than SRLTLE if and only if $\lambda_{\max}[\tilde{F}_{Opt}R\tilde{F}'_{Opt}(Z_{k,d}RZ'_{k,d})^{-1}] < 1$ and $\delta'_{Opt}[Z_{k,d}RZ'_{k,d} - \tilde{F}_{Opt}R\tilde{F}'_{Opt} + \beta'(Z_{k,d} - I)'(Z_{k,d} - I)\beta]^{-1}\delta_{Opt} \leq 1$.

3.2 SROLE versus OGLE

Consider

$$\begin{aligned}
&MSEM(\hat{\beta}_{OGLE}) - MSEM(\hat{\beta}_{SROLE}) \\
&\quad = \{\tilde{J}_{(i)}C^{-1}\tilde{J}'_{(i)} + (\tilde{J}_{(i)} - I)\beta\beta'(\tilde{J}_{(i)} - I)'\} \\
&\qquad - \{\tilde{F}_{Opt}R\tilde{F}'_{Opt} + (\tilde{F}_{Opt} - I)\beta\beta'(\tilde{F}_{Opt} - I)'\} \\
&\quad = \{\tilde{J}_{(i)}C^{-1}\tilde{J}'_{(i)} - \tilde{F}_{Opt}R\tilde{F}'_{Opt}\} \\
&\qquad + \{(\tilde{J}_{(i)} - I)\beta\beta'(\tilde{J}_{(i)} - I)' - (\tilde{F}_{Opt} - I)\beta\beta'(\tilde{F}_{Opt} - I)'\}. \qquad (3.4)
\end{aligned}$$

Now consider

$$\begin{aligned}
D(\hat{\beta}_{OGLE}) - D(\hat{\beta}_{SROLE}) &= \tilde{J}_{(i)}C^{-1}\tilde{J}'_{(i)} - \tilde{F}_{Opt}R\tilde{F}'_{Opt} \\
&= D^{**}. \qquad (3.5)
\end{aligned}$$

Note that, since C^{-1} and R are positive definite matrices, $\tilde{J}_{(i)}C^{-1}\tilde{J}'_{(i)}$ and $\tilde{F}_{Opt}R\tilde{F}'_{Opt}$ are positive definite matrices (see Rao and Toutenburg 1995, p. 366). Consequently, by Lemma 4, if $\lambda_{\max}(\tilde{F}_{Opt}R\tilde{F}'_{Opt}(\tilde{J}_{(i)}C^{-1}\tilde{J}'_{(i)})^{-1}) < 1$ then D^{**} is a positive definite matrix, where $\lambda_{\max}[\tilde{F}_{Opt}R\tilde{F}'_{Opt}(\tilde{J}_{(i)}C^{-1}\tilde{J}'_{(i)})^{-1}] < 1$ is the largest eigen value of

$\tilde{F}_{Opt}R\tilde{F}'_{Opt}(\tilde{J}_{(i)}C^{-1}\tilde{J}'_{(i)})^{-1}$. Further, by Lemma 5, $MSE(\hat{\beta}_{OGLE})-MSE(\hat{\beta}_{SROLE})$ is non-negative definite if $\delta'_{Opt}[D^{**}+\delta'_J\delta_J]^{-1}\delta_{Opt}\leq 1$, where $\delta_{Opt}=(\tilde{F}_{Opt}-I)\beta$ and $\delta_J=(\tilde{J}_{(i)}-I)\beta$.
Based on the above arguments, we state the following theorem.

Theorem 2 *When* $\lambda_{\max}[\tilde{F}_{Opt}R\tilde{F}'_{Opt}(\tilde{J}_{(i)}C^{-1}\tilde{J}'_{(i)})^{-1}] < 1$, *the estimator SROLE is superior than OGLE if and only if* $\delta'_{Opt}[D^{**}+\delta'_J\delta_J]^{-1}\delta_{Opt}\leq 1$.

4 Simulation study

A Monte Carlo simulation study is performed to examine the performance of the proposed optimal estimator SROLE with the existing estimators MLE, SRMLE, SRRMLE, SRLMLE, SRAURLE, SRAULLE, SRLTLE, and OGLE by considering different levels of multicollinearity; $\rho = 0.95$, 0.99, and 0.999. The Scalar Mean Square Error (SMSE) is considered for the comparison of estimators.
According to McDonald and Galarneau (1975) and Kibria (2003), we generate the explanatory variables as follows:

$$x_{ij}=(1-\rho^2)^{1/2}z_{ij}+\rho z_{i,p+1}, i=1,2,\ldots,n, \quad j=1,2,\ldots,p, \tag{4.1}$$

where z_{ij} are independent standard normal pseudo- random numbers and ρ is specified so that the theoretical correlation between any two explanatory variables is given by ρ^2. Two sets of explanatory variables having $p=2$ and $p=4$ are generated using (4.1). The dependent variable y_i in (1.1) is obtained from the Bernoulli(π_i) distribution where $\pi_i=\frac{\exp(x_i'\beta)}{1+\exp(x_i'\beta)}$, and three different sample sizes; $n = 20$, 50, and 100 are considered for the study. Following Mansson and Shukur (2011) and Şiray et al. (2015), the parameter values of $\beta_1,\beta_2,\ldots,\beta_p$ are chosen so that $\sum_{j=1}^{p}\beta_j^2=1$ and $\beta_1=\beta_2=\ldots=\beta_p$. For the simulation, according to Şiray et al. (2015), we choose the same restrictions $H=[1,-1]$, $h=0$, $\Psi=1$ in the case of $p=2$ and

$$H=\begin{pmatrix}1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1\end{pmatrix},\quad h=\begin{pmatrix}0\\0\\0\end{pmatrix},\ \text{and}\quad \Psi=\begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}. \tag{4.2}$$

in the case of $p = 4$. The selection of the above stochastic restrictions will give researchers to understand the comparisons of results with other studies which have already being done using the same restrictions. The simulation is repeated 1000 times by generating new pseudo-random numbers and the simulated SMSE values of the estimators are obtained using the following equation.

$$S\hat{M}SE(\hat{\beta})=\frac{1}{1000}\sum_{r=1}^{1000}(\hat{\beta}_r-\beta)'(\hat{\beta}_r-\beta), \tag{4.3}$$

where $\hat{\beta}_r$ is any estimator considered in the rth. simulation.

Since the biasing parameters k, d of the estimators SRRMLE, SRLMLE, SRAURLE, SRAULLE, and SRLTLE varies within the ranges of $k \geq 0$ and $0 \leq d \leq 1$, it is essential to identify the optimal values of k, and d. As such, following Özkale (2015) and van Howelingen and Sauerbrei (2013), 5-fold cross validation method was applied for each set of sample data when $\rho = 0.95$, 0.99, and 0.999 and $n = 20, 50, 100$. In this approach, the total cross validation error, which is the sum of prediction errors of all possible test data sets, is obtained for the different values of k, and d within the ranges $k \geq 0$ and $0 \leq d \leq 1$, respectively. The values of k, d that minimizes the total cross validation error will be the optimal. The optimal values of k, d of the estimators SRRMLE, SRLMLE, SRAURLE, SRAULLE, and SRLTLE corresponding to each case of $\rho = 0.95$, 0.99, and 0.999 and $n = 20, 50, 100$ are reported in Tables 1, 2, 3, 4, 5 and 6 in Appendix A.

The estimated scalar mean square errors are reported in Tables 7, 8, 9, 10, 11 and 12 in Appendix A and displayed in Figs. 1, 2, 3, 4, 5 and 6 in Appendix B. Note that, since the SMSE of MLE is very high compared to other estimators, we skip the MLE in the construction of Figs. 1, 2, 3, 4, 5 and 6. It can be observed from Figs. 1, 2, 3, 4, 5 and 6 and Tables 7, 8, 9, 10, 11 and 12 that the proposed optimal estimator SROLE outperforms the estimators SRMLE, SRRMLE, SRLMLE, SRAURLE, SRAULLE, SRLTLE, and OGLE in the scalar mean square error sense with respect to all the values of $\rho = 0.95$, 0.99, and 0.999 and $n = 20, 50, 100$. Since MLE, SRMLE, OGLE, and SROLE do not depend on the parameters k and d, the SMSE values of these estimators are constant for different choice of k, d. Further, for all cases, the performance of MLE is worst since the SMSE values are higher in comparison to the other estimators, and SRMLE shows the second worst performance compared to other estimators. In general, the SMSE of all the estimators increases when the degree of collinearity increases. Moreover, the SRLTLE and OGLE respectively show the second and third best performances compared to all the other estimators with respect to all ρ, n, and p values considered in this study. It can be further noted that the SMSE values of the estimators decreases whenever n increases.

Further, considering two explanatory variables ($p = 2$), among the estimators SRRMLE, SRLMLE, SRAURLE, and SRAULLE; SRAURLE performs better when $\rho = 0.95$, and for all sample sizes $n = 20, 50, 100$. However, when $\rho = 0.99$, and 0.999, SRLMLE performs well compared to SRRMLE, SRAURLE, and SRAULLE, regardless of the values of n. For four explanatory variables ($p = 4$), SRLMLE outperforms the estimators SRRMLE, SRAURLE, and SRAULLE with respect all the values ρ and n, except the case of $\rho = 0.95$ and $n = 100$.

5 A real data example

In this section, we use the data set, which has been previously analyzed by Asar and Genç (2016), Wu and Asar (2016), Varathan and Wijekoon (2016b) and among others to show the performance of the proposed estimator with the existing estimators. The data set consists the information of hundred municipalities of Sweden with four predictor variables and a response variable. The response variable is the Net population change (y) which takes the value 1 if there is an increase in the population and 0 otherwise, and the predictor variables are Population ($x1$), Number of unemployed people ($x2$), Number of newly constructed buildings ($x3$), and Number of bankrupt firms ($x4$). The condition number being a measure of multicollinearity is obtained as 188, indicates the existence of severe multicollinearity in the data set. Further, Variance inflation factors (VIF) of the data are 488.17, 344.26, 44.99, and 50.71, which also confirm the multicollinearity in the data. Further, since we have no prior information for this numerical example, we use the same prior information as used in the simulation study. The SMSE values of MLE, SRMLE, SRRMLE, SRLMLE, SRAURLE, SRAULLE, SRLTLE, OGLE, and SROLE are calculated for the optimal values of biasing parameters k, d obtained through the cross validation. The optimal values of k, d are given in the Table 13 and the SMSE values are given in the Table 14 in Appendix A. Results, reveal that the proposed estimator SROLE outperforms all the other existing estimators considered in this study. Further, as we noted in the simulation study, SRLTLE and OGLE respectively give the second and third best performances with respect to the scalar mean square sense.

6 Concluding remarks

In this article, a new optimal estimator is proposed for logistic regression model when the prior information is available in the form of stochastic linear restrictions. The superiority conditions for the proposed estimator with the existing estimators SRMLE, SRRMLE, SRLMLE, SRAURLE, SRAULLE, SRLTLE, and OGLE are derived with respect to mean square error matrix criterion. Further, from the simulation study it was noticed that, in general the proposed optimal estimator SROLE performed well compared to other existing estimators in the scalar mean square error sense. Finally, a real data example is examined to verify the theoretical findings.

Appendix A

See Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14.

Table 1 The optimal values of k, d for different ρ values when $p = 2$ and $n = 20$

	SRRMLE	SRAURLE	SRLMLE	SRAULLE	SRLTLE
$\rho = 0.95$	$k = 0.098$	$k = 18.745$	$d = 0.963$	$d = 0.960$	$k = 0.060, d = 0.130$
$\rho = 0.99$	$k = 1.875$	$k = 15.316$	$d = 0.587$	$d = 0.581$	$k = 0.081, d = 0.001$
$\rho = 0.999$	$k = 4.105$	$k = 9.803$	$d = 0.562$	$d = 0.561$	$k = 0.009, d = 0.032$

Table 2 The optimal values of k, d for different ρ values when $p = 2$ and $n = 50$

	SRRMLE	SRAURLE	SRLMLE	SRAULLE	SRLTLE
$\rho = 0.95$	$k = 0.108$	$k = 25.635$	$d = 0.845$	$d = 0.834$	$k = 0.090, d = 0.280$
$\rho = 0.99$	$k = 2.345$	$k = 24.833$	$d = 0.632$	$d = 0.603$	$k = 0.110, d = 0.003$
$\rho = 0.999$	$k = 3.654$	$k = 13.350$	$d = 0.629$	$d = 0.620$	$k = 0.011, d = 0.001$

Table 3 The optimal values of k, d for different ρ values when $p = 2$ and $n = 100$

	SRRMLE	SRAURLE	SRLMLE	SRAULLE	SRLTLE
$\rho = 0.95$	$k = 0.203$	$k = 29.980$	$d = 0.379$	$d = 0.479$	$k = 0.320, d = 0.150$
$\rho = 0.99$	$k = 4.758$	$k = 21.746$	$d = 0.647$	$d = 0.641$	$k = 0.140, d = 0.020$
$\rho = 0.999$	$k = 5.078$	$k = 5.447$	$d = 0.882$	$d = 0.879$	$k = 0.020, d = 0.003$

Table 4 The optimal values of k, d for different ρ values when $p = 4$ and $n = 20$

	SRRMLE	SRAURLE	SRLMLE	SRAULLE	SRLTLE
$\rho = 0.95$	$k = 5.735$	$k = 12.405$	$d = 0.205$	$d = 0.197$	$k = 0.630, d = 0.060$
$\rho = 0.99$	$k = 2.340$	$k = 14.325$	$d = 0.285$	$d = 0.265$	$k = 0.270, d = 0.260$
$\rho = 0.999$	$k = 1.530$	$k = 9.420$	$d = 0.365$	$d = 0.352$	$k = 2.800, d = 0.850$

Table 5 The optimal values of k, d for different ρ values when $p = 4$ and $n = 50$

	SRRMLE	SRAURLE	SRLMLE	SRAULLE	SRLTLE
$\rho = 0.95$	$k = 7.845$	$k = 17.651$	$d = 0.275$	$d = 0.243$	$k = 1.100, d = 0.090$
$\rho = 0.99$	$k = 4.325$	$k = 19.265$	$d = 0.361$	$d = 0.360$	$k = 0.420, d = 0.540$
$\rho = 0.999$	$k = 2.165$	$k = 11.305$	$d = 0.472$	$d = 0.468$	$k = 2.700, d = 0.920$

Table 6 The optimal values of k, d for different ρ values when $p = 4$ and $n = 100$

	SRRMLE	SRAURLE	SRLMLE	SRAULLE	SRLTLE
$\rho = 0.95$	$k = 11.900$	$k = 28.367$	$d = 0.487$	$d = 0.475$	$k = 1.800, d = 0.110$
$\rho = 0.99$	$k = 10.981$	$k = 31.112$	$d = 0.567$	$d = 0.560$	$k = 0.560, d = 0.840$
$\rho = 0.999$	$k = 4.555$	$k = 1.616$	$d = 0.892$	$d = 0.854$	$k = 2.100, d = 0.850$

Table 7 The estimated SMSE values for different ρ values when $p = 2$ and $n = 20$

	MLE	SRMLE	SRRMLE	SRLMLE	SRAULLE	SRAURLE	SRLTLE	OGLE	SROLE
$\rho = 0.95$	2.5873	0.5317	0.5044	0.5096	0.5311	0.3735	0.2270	0.2273	0.2261
$\rho = 0.99$	12.1738	0.5944	0.5341	0.2901	0.4664	0.5000	0.2420	0.2636	0.2388
$\rho = 0.999$	120.0946	0.6124	0.5486	0.2632	0.4420	0.5541	0.2510	0.2602	0.2418

Table 8 The estimated SMSE values for different ρ values when $p = 2$ and $n = 50$

	MLE	SRMLE	SRRMLE	SRLMLE	SRAULLE	SRAURLE	SRLTLE	OGLE	SROLE
$\rho = 0.95$	0.9377	0.3624	0.3429	0.3172	0.3590	0.2278	0.2151	0.2201	0.2026
$\rho = 0.99$	4.4163	0.4899	0.4354	0.2620	0.4140	0.4267	0.2430	0.2495	0.2334
$\rho = 0.999$	43.5787	0.5365	0.5202	0.2640	0.4152	0.5203	0.2521	0.2583	0.2413

Table 9 The estimated SMSE values for different ρ values when $p = 2$ and $n = 100$

	MLE	SRMLE	SRRMLE	SRLMLE	SRAULLE	SRAURLE	SRLTLE	OGLE	SROLE
$\rho = 0.95$	0.4511	0.2518	0.2326	0.1834	0.2359	0.1247	0.1767	0.1786	0.1717
$\rho = 0.99$	2.1234	0.4238	0.3428	0.2544	0.3792	0.3412	0.2420	0.2485	0.2246
$\rho = 0.999$	20.9504	0.5090	0.4937	0.4052	0.4967	0.4969	0.2449	0.2542	0.2404

Table 10 The estimated SMSE values for different ρ values when $p = 4$ and $n = 20$

	MLE	SRMLE	SRRMLE	SRLMLE	SRAULLE	SRAURLE	SRLTLE	OGLE	SROLE
$\rho = 0.95$	8.5734	1.7342	1.2126	0.3763	0.8248	1.2177	0.2550	0.2957	0.2536
$\rho = 0.99$	41.7881	2.3064	2.0725	0.3204	0.7856	2.0638	0.2657	0.2985	0.2593
$\rho = 0.999$	415.5559	2.5304	2.4918	0.3947	0.9608	2.4868	0.2725	0.3415	0.2606

Table 11 The estimated SMSE values for different ρ values when $p = 4$ and $n = 50$

	MLE	SRMLE	SRRMLE	SRLMLE	SRAULLE	SRAURLE	SRLTLE	OGLE	SROLE
$\rho = 0.95$	2.8555	1.1585	0.6150	0.5037	0.8917	0.6161	0.2600	0.2855	0.2354
$\rho = 0.99$	13.9155	1.9706	1.5627	0.4718	1.0579	1.5615	0.2820	0.2924	0.2560
$\rho = 0.999$	138.3692	2.4493	2.3764	0.5910	1.3296	2.3766	0.2958	0.3263	0.2601

Table 12 The estimated SMSE values for different ρ values when $p = 4$ and $n = 100$

	MLE	SRMLE	SRRMLE	SRLMLE	SRAULLE	SRAURLE	SRLTLE	OGLE	SROLE
$\rho = 0.95$	1.3585	0.7743	0.3141	0.5498	0.7349	0.3124	0.2231	0.2675	0.2059
$\rho = 0.99$	6.6207	1.6256	1.0993	0.8110	1.3536	1.0994	0.2629	0.2879	0.2496
$\rho = 0.999$	65.8360	2.3684	2.2346	1.9073	2.3186	2.2361	0.3738	0.3951	0.2595

Table 13 The optimal values of k, d for real data example

SRRMLE	SRAURLE	SRLMLE	SRAULLE	SRLTLE
$k = 1.564$	$k = 6.632$	$d = 0.625$	$d = 0.602$	$k = 0.150, d = 0.540$

Table 14 The SMSE values for real data example

MLE	SRMLE	SRRMLE	SRLMLE	SRAULLE	SRAURLE	SRLTLE	OGLE	SROLE
0.5661	0.3819	0.2248	0.3009	0.3713	0.2941	0.1958	0.2130	0.1856

Appendix B

See Figs. 1, 2, 3, 4, 5 and 6.

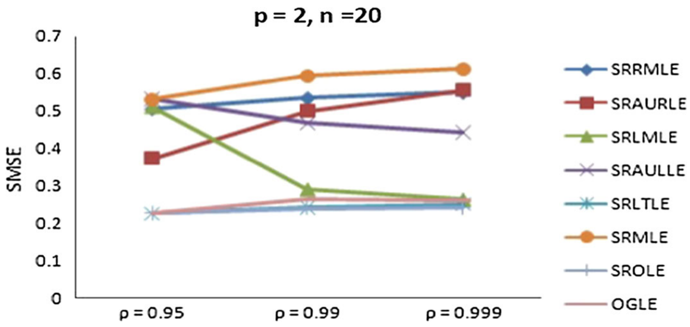

Fig. 1 Estimated SMSE values for SRMLE, SRRMLE, SRLMLE, SRAULLE, SRAURLE, SRLTLE, OGLE, and SROLE for p = 2, n = 20

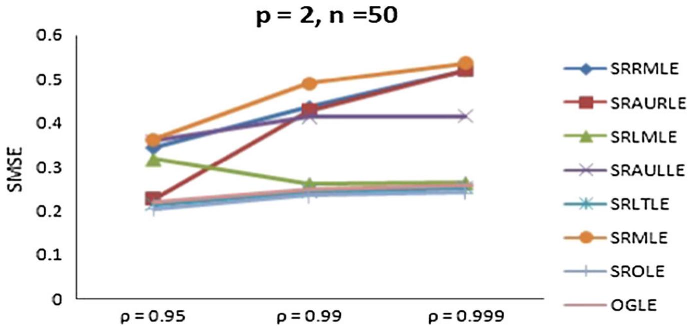

Fig. 2 Estimated SMSE values for SRMLE, SRRMLE, SRLMLE, SRAULLE, SRAURLE, SRLTLE, OGLE, and SROLE for p = 2, n = 50

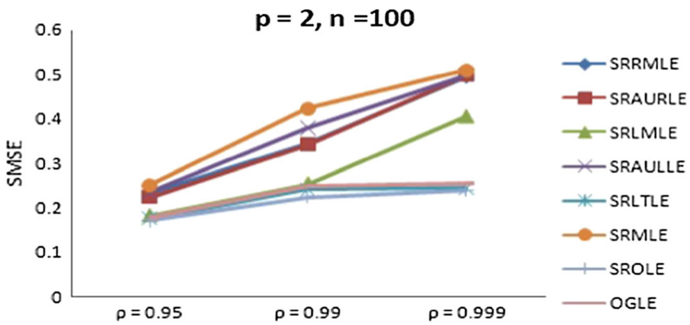

Fig. 3 Estimated SMSE values for SRMLE, SRRMLE, SRLMLE, SRAULLE, SRAURLE, SRLTLE, OGLE, and SROLE for p = 2, n = 100

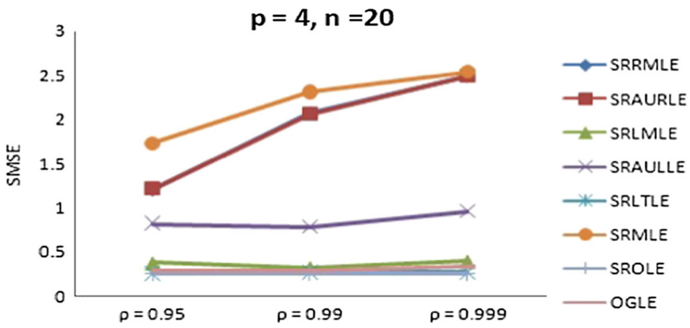

Fig. 4 Estimated SMSE values for SRMLE, SRRMLE, SRLMLE, SRAULLE, SRAURLE, SRLTLE, OGLE, and SROLE for p = 4, n = 20

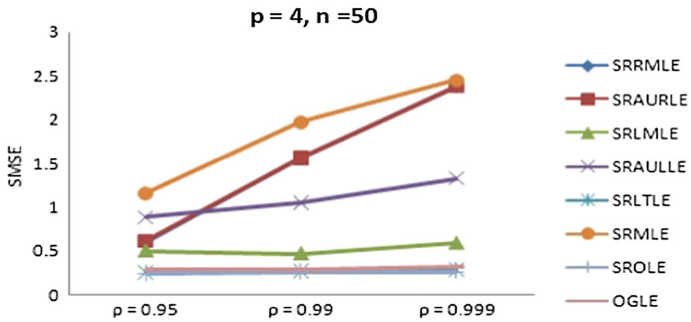

Fig. 5 Estimated SMSE values for SRMLE, SRRMLE, SRLMLE, SRAULLE, SRAURLE, SRLTLE, OGLE, and SROLE for p = 4, n = 50

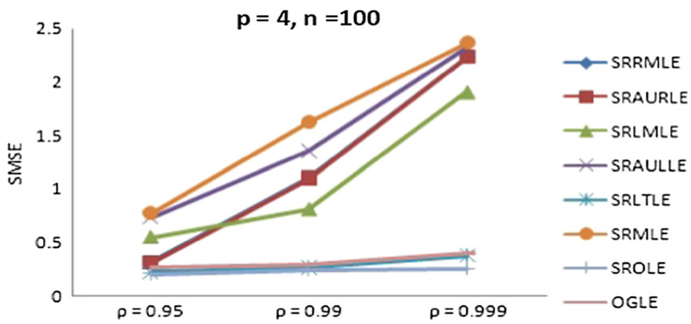

Fig. 6 Estimated SMSE values for SRMLE, SRRMLE, SRLMLE, SRAULLE, SRAURLE, SRLTLE, OGLE, and SROLE for p = 4, n = 100

References

Aguilera AM, Escabias M, Valderrama MJ (2006) Using principal components for estimating logistic regression with high-dimensional multicollinear data. Comput. Stat. Data Anal. 50:1905–1924

Asar Y, Arashi M, Wu J (2017a) Restricted ridge estimator in the logistic regression model. Commun Stat Simul Comput 46(8):6538–6544

Asar Y, Erişoğlu M, Arashi M (2017b) Developing a restricted two-parameter Liu-type estimator: a comparison of restricted estimators in the binary logistic regression model. Commun Stat Theory Method 46(14):6864–6873

Asar Y, Genç A (2016) New shrinkage parameters for the Liu-type logistic estimators. Commun Stat Simul Comput 45(3):1094–1103

Cox D (1958) The regression analysis of binary sequences (with discussion). J R Stat Soc B 20(2):215–242

Duffy DE, Santner TJ (1989) On the small sample prosperities of norm-restricted maximum likelihood estimators for logistic regression models. Commun Stat Theory Methods 18:959–980

Inan D, Erdogan BE (2013) Liu-type logistic estimator. Commun Stat Simul Comput 42:1578–1586

Kibria BMG (2003) Performance of some new ridge regression estimators. Commun Stat Theory Methods 32:419–435

Mansson G, Kibria BMG, Shukur G (2012) On Liu estimators for the logit regression model. The Royal Institute of Techonology, Centre of Excellence for Science and Innovation Studies (CESIS), Sweden, Paper No. 259

Mansson K, Shukur G (2011) On ridge parameters in logistic regression. Commun Stat Theory Methods 40:3366–3381

McDonald GC, Galarneau DI (1975) A Monte Carlo evaluation of some ridge type estimators. J Am Stat Assoc 70:407–416

Nagarajah V, Wijekoon P (2015) Stochastic restricted maximum likelihood estimator in logistic regression model. Open J Stat 5:837–851. https://doi.org/10.4236/ojs.2015.57082

Newhouse JP, Oman SD (1971) An evaluation of ridge estimators. RAND Corporation, Santa Monica

Nja ME, Ogoke UP, Nduka EC (2013) A new logistic ridge regression estimator using exponentiated response function. J Stat Econ Methods 2(4):161–171

Özkale MR (2015) Predictive performance of linear regression models. Stati Pap 56(2):531–67

Rao CR, Toutenburg H, Shalabh HC (2008) Linear models and generalizations. Springer, Berlin

Rao CR, Toutenburg H (1995) Linear models : least squares and alternatives, 2nd edn. Springer, New York

Schaefer RL, Roi LD, Wolfe RA (1984) A ridge logistic estimator. Commun Stat Theory Methods 13:99–113

Şiray GU, Toker S, Kaçiranlar S (2015) On the restricted Liu estimator in logistic regression model. Commun Stat Simul Comput 44:217–232

Trenkler G, Toutenburg H (1990) Mean square error matrix comparisons between biased estimators? An overview of recent results. Stat Pap 31:165–179. https://doi.org/10.1007/BF02924687

van Howelingen HC, Sauerbrei W (2013) Cross-validation, shrinkage and variable selection in linear regression revisited. Open J Stat 03(02):79–102

Varathan N, Wijekoon P (2016a) On the restricted almost unbiased ridge estimator in logistic regression. Open J Stat 6:1076–1084. https://doi.org/10.4236/ojs.2016.66087

Varathan N, Wijekoon P (2016b) Ridge estimator in logistic regression under stochastic linear restriction. Br J Math Comput Sci 15(3):1. https://doi.org/10.9734/BJMCS/2016/24585

Varathan N, Wijekoon P (2016c) Logistic Liu estimator under stochastic linear restrictions. Stat Pap. https://doi.org/10.1007/s00362-016-0856-6

Varathan N, Wijekoon P (2017) A stochastic restricted almost unbiased ridge estimator in logistic regression. Proceedings of the iPURSE, University of Peradeniya, p 36. Accessed 24 Nov 2017

Varathan N, Wijekoon P (2018a) Optimal generalized logistic estimator. Commun Stat Theory Methods 47(2):463–474

Varathan N, Wijekoon P (2018b) Liu-type logistic estimator under stochastic linear restrictions. Ceylon J Sci 47(1):21–34. https://doi.org/10.4038/cjs.v47i1.7483

Varathan N, Wijekoon P (2018c) An improved stochastic restricted almost unbiased Liu-estimator in logistic regression. J Mod Appl Stat Methods, Accepted

Wu J (2016) Modified restricted Liu estimator in logistic regression model. Comput Stat 31(4):1557–1567

Wu J, Asar Y (2016) On almost unbiased ridge logistic estimator for the logistic regression model. Hacet J Math Stat 45(3):989–998. https://doi.org/10.15672/HJMS.20156911030

Xinfeng C (2015) On the almost unbiased ridge and Liu estimator in the logistic regression model. In: International conference on social science, education management and sports education. Atlantis Press, Paris, pp 1663–1665