Limitations on Numerical Iterative Solution of Elliptic Two-Dimensional Partial Differential Equations

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Abstract

Numerical iterative methods are applied for the solution of two dimensional Elliptic partial differential equations such as Laplace's and Poisson's equations. These kinds of differential equations have specific applications models of physics and engineering. The distinct approximation of the two equations is founded upon the theory of finite difference. In this work, the approximation of five point's scheme of finite difference method is used for the equations of Laplace and Poisson to get linear system of equations. The solution of these Dirichlet boundary is discussed by finite difference method. An elliptic PDE transforms the PDE into a system of algebraic equations whose coefficient matrix has a tri-diagonal block format, using the finite difference method. Numerical iterative methods such as Jacobi method and Gauss-Seidel method are used to solve the resulting finite difference approximation with boundary conditions.

Keywords: iterative solution, elliptic partial differential equations, boundary conditions

Introduction

The main aim of the research is finding limitation on numerical solution of Elliptic PDEs by using iterative methods. The majority of problems cannot be analytically solved, so it would be very useful to find good approximate solutions using numerical methods. Discretizing the elliptic Poisson equation with homogeneous Dirichlet boundary conditions by the finite difference method results in a system of linear equations with a large, sparse, highly structured system matrix. Idea of finite difference method is to discretize the partial differential equation by replacing partial derivatives with their approximation that is finite differences. The PDE is transformed using this approach into a series of linear, simultaneous equations. Which is written in the matrix equation and then the solution is obtained by solving the matrix equation or the solution can be obtained iteratively by solving simultaneous equations.

Background

Together with Dirichlet boundary conditions, numerical techniques were introduced to solve a two-dimensional Poisson equation. Specifically, Finite difference method and Finite element methods are used for the purpose of numerical solution. The implementation of the solutions is achieved using the worksheet or spreadsheet of Microsoft Office Excel. The numerical solutions obtained by these two approaches are often graphically compared to each other in two and three dimensions.(Patil and Prasad, 2013)

Numerical techniques were introduced to solve a two-dimensional Laplace equation with Dirichlet boundary conditions. Using spreadsheets, they used finite difference and finite element techniques to solve two-dimensional Laplace equations with Dirichlet boundary conditions (Mebrate, 2015).

Classification of Partial Differential Equation

The general form for linear second-order PDEs with two independent variables x, y is,

$$\mathrm{A}u_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0 \quad (2.1)$$

Where the coefficients A, B, C, D, E, F and G are constants or are functions of the independent variables x and y.
The classification is based on the discriminative sign $B^2 - 4AC$ as follows.

The PDEs are considered Elliptic if $B^2 - 4AC < 0$.

Finite difference approximation of second-order partial derivatives

$$u_{xx}(x_i, y_j) = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \quad (2.2)$$

$$u_{yy}(x_i, y_j) = \frac{u_{j-1} - 2u_j + u_{j+1}}{m^2} \quad (2.3)$$

Difference Schemes for the Elliptic PDE's

Defined over $\Omega = \{(x, y)|0 < x, y < 1\}$ respectively with Dirichlet boundary conditions

$$u(x, y) = g(x, y), \text{ for all } (x, y) \in \partial\Omega \quad (2.4)$$

The finite difference approximation of the equation of Laplace's and Poisson's at the point i, j has the following form.

If we consider distance $h = m$,

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0 \quad (2.5)$$

and

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{ij} \quad (2.6)$$

The five-point finite difference approximation for the equation of Laplace's and Poisson's is called equations (2.5) and (2.6) respectively.

Equation (2.5) and (2.6) are assembled into a linear system of equations.

If the various equations are taken in the order of the point, the coefficient matrix A is

$$A = \left[\begin{array}{c|c|c} B & C & 0 \\ \hline C & B & C \\ \hline 0 & C & B \end{array}\right]$$

Where

$$B = \begin{bmatrix} -4 & 1 & . & . & . & . & 0 \\ 1 & -4 & 1 & . & . & . & . \\ . & 1 & -4 & 1 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & -4 & 1 \\ 0 & . & . & . & . & 1 & -4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & . & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & . \\ . & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & 1 & 0 \\ 0 & . & . & . & . & 0 & 1 \end{bmatrix}$$

Methodology

To solve linear algebraic system $Au = b$, (3.1)

Obtained from the discretization of an Elliptic partial differential equation, where A is large definite $n \times n$ matrix that is sparse and typically positive.

Consider the splitting

$$A = M - N \tag{3.2}$$

We may write the existing system equation (3.1) is,

$$(M - N)u = b$$

The iterative form is

$$Mu^{(k+1)} = Nu^{(k)} + b \tag{3.3}$$

Which is equivalently as

$$u^{(k+1)} = (M^{-1}N)u^{(k)} + M^{-1}b \tag{3.4}$$

3.1 Jacobi and Gauss-Seidel Method

Consider a linear system $Ax = b$, the equation (3.1) can be written in the form of

$$(D - L - U)u = b \tag{3.5}$$
$$u = D^{-1}(L + U)u + D^{-1}b \tag{3.6}$$

And consider the iteration

$$u^{(k+1)} = D^{-1}(L + U)u^{(k)} + D^{-1}b \tag{3.7}$$

If the equations (3.6) or (3.7) are used to solve the finite difference equation method for the Laplace's and Poisson's equation, we obtain the formula of the Jacobi iteration.

$$u_{i,j}^{(k+1)} = \frac{1}{4}\left[u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)}\right] \tag{3.8}$$

and

$$u_{i,j}^{(k+1)} = \frac{1}{4}\left[u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)} - h^2 f_{ij}\right] \qquad (3.9)$$

consecutively. Solution updates at (i,j) are measured at their four adjacent points as a weighted average of solutions.

Matrix form of Gauss-Seidel method is

$$(D-L)u^{(k+1)} = Uu^{(k)} + b \qquad (3.10)$$

or

$$u^{(k+1)} = (D-L)^{-1}Uu^{(k)} + (D-L)^{-1}b \qquad (3.11)$$

If the equations (3.11) are used to resolve the scheme of finite difference equations for the equation of Laplace's and Poisson's, we obtain

$$u_{i,j}^{(k+1)} = \frac{1}{4}\left[u_{i-1,j}^{(k+1)} + u_{i,j-1}^{(k+1)} + u_{i+1,j}^{(k)} + u_{i,j+1}^{(k)}\right] \qquad (3.12)$$

and

$$u_{i,j}^{(k+1)} = \frac{1}{4}\left[u_{i-1,j}^{(k+1)} + u_{i,j-1}^{(k+1)} + u_{i+1,j}^{(k)} + u_{i,j+1}^{(k)} - h^2 f_{ij}\right] \qquad (3.13)$$

Results and Discussion

Problem 1

Laplace's equation in two dimension $u_{xx} + u_{yy} = 0$ $S = \{(x,y)|0 \le x \le 1, 0 \le y \le 1\}$ in the unit square with boundary condition $u(0,y) = 50, u(x,1) = 100, u(x,0) = 300, u(1,y) = 200$

Problem 2

Consider the Poisson's equation,

$$u_{xx} + u_{yy} = -2 \quad , \quad S = \{(x,y)|0 \le x \le 1, 0 \le y \le 1\}$$

From the five point difference discretization, the generated linear scheme of the above model problems is resolved.

According to the fixed iterative vector, the iterative algorithm of the above methods with different values of step-size values $h = \frac{1}{10}$, $h = \frac{1}{20}$, $h = \frac{1}{40}$ has been computed numerically.

These illustrations show that iterative approach of Gauss Seidel requires less iteration than the techniques of Jacobi.

Approximate solutions of to $h = \frac{1}{40}$ problems 1 and 2 are shown in Figures (4.1), (4.2) and Figures (4.3), (4.4) respectively.

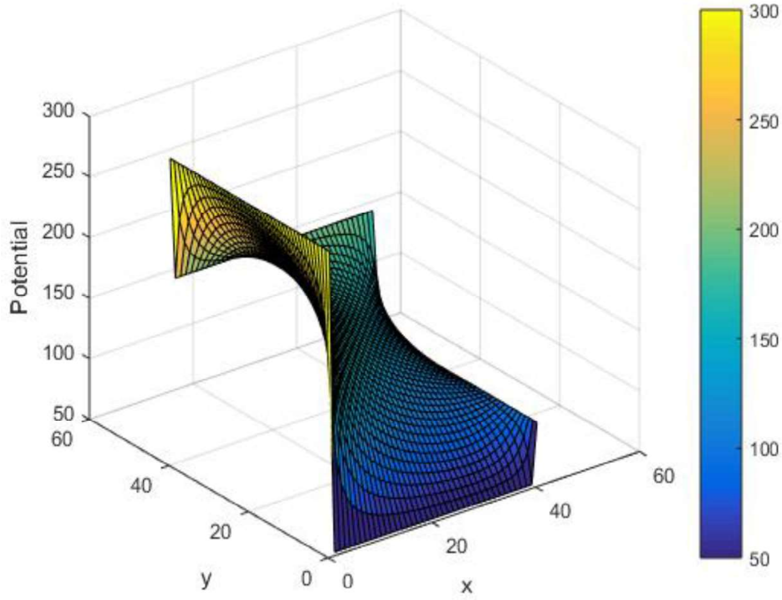

Figure 4.1: Surface plot of potential distribution through Jacobi Iteration Method, For problem 1, $h = 1/40$

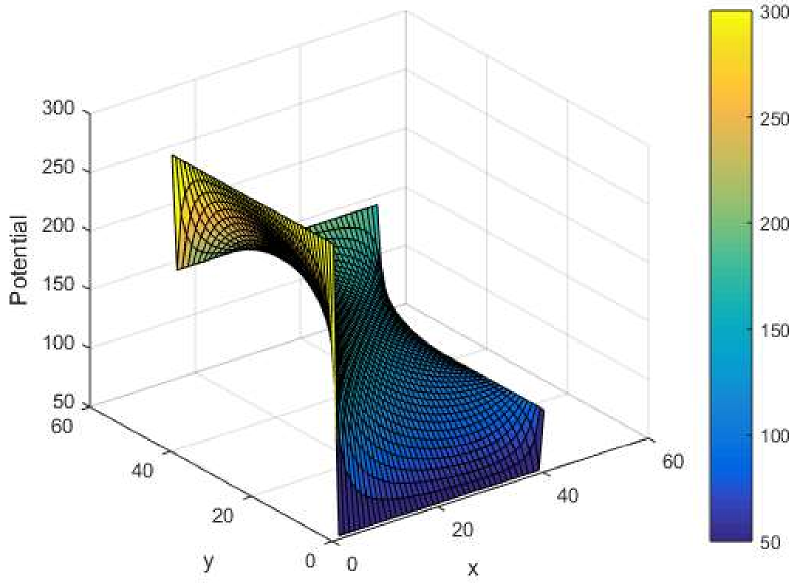

Figure 4.2: Surface plot of potential distribution through Gauss-Seidel Iteration Method, For problem 1, $h = 1/40$

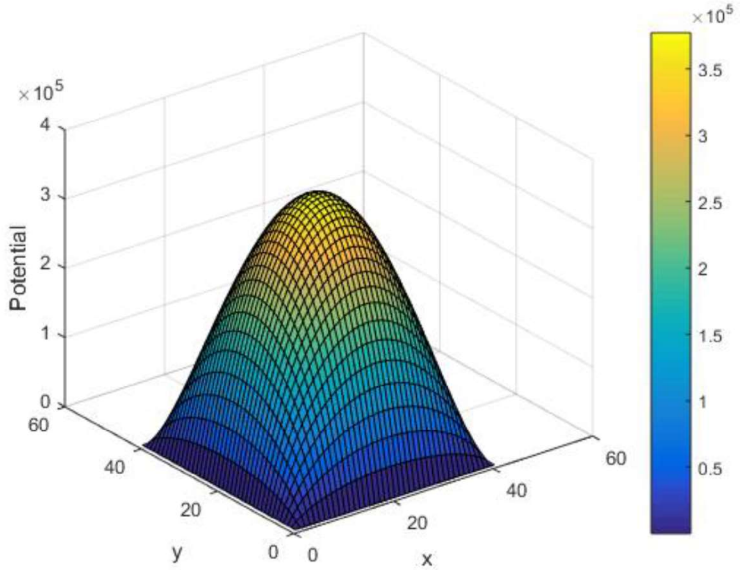

Figure 4.3: Surface plot of potential distribution through Jacobi Iteration Method, For problem 2, $\mathrm{h} = 1/40$

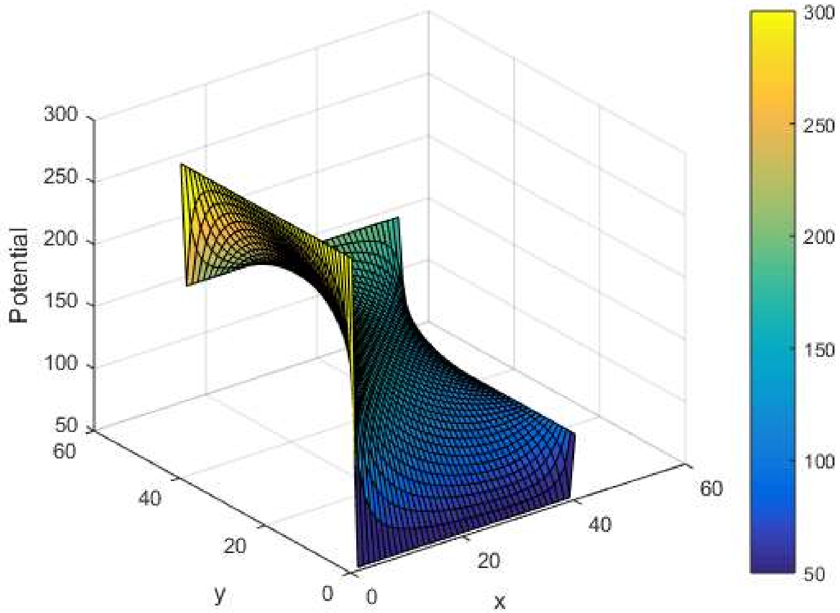

Figure 4.4: Surface plot of potential distribution through Gauss-Seidel Iteration Method for $\mathrm{h} = 1/40$ for problem 2

Table 4.1: Total number of iterations with different values of $h = \frac{1}{10}, h = \frac{1}{20}, h = \frac{1}{40}$ for Jacobi and Gauss Seidel methods problem 1

	Jacobi	Gauss-seidel
h	**Number of iteration**	**Number of iteration**
1/10	222	119
1/20	775	416
1/40	2640	1433

Table 4.2: Total number of iterations with different values of $h = \frac{1}{10}, h = \frac{1}{20}, h = \frac{1}{40}$ for Jacobi and Gauss Seidel methods problem 2

	Jacobi	Gauss-seidel
h	**Number of iteration**	**Number of iteration**
1/10	261	119
1/20	1147	416
1/40	5030	1433

Conclusion

In order to compare the efficacy of the simple iterative methods to explore the limitations on numerical solutions, two practical problems were solved for different step-sizes h. This algorithm is more user friendly to obtain approximate solutions of elliptic Pde's. Having better convergece iterative solution by Gauss-Seidel method and the error is occuring due to the elimination of order of the step-sizes.We analysed the model problems with several step-sizes with the help of boundary conditions.

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