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INVARIANT APPROXIMATION PROPERTY FOR DIRECT PRODUCT WITH A FINITE GROUP

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ABSTRACT. We will study the invariant approximation property in various contexts. An interesting question, which we will address next is the behavior of this property with respect to group extensions. To prepare for that we first study a relationship of uniform Roe algebras attached to coarsely equivalent metric spaces in the following case. Let X be a bounded geometry metric space and assume that there is a bijective coarse equivalence

$$\phi : X \longrightarrow Y \times N,$$

where N is a finite metric space. Then there is an isomorphism

$$\begin{aligned} C_u^*(X) &\cong C_u^*(Y) \otimes C_u^*(N) \\ &\cong C_u^*(Y) \otimes M_n(\mathbb{C}), \end{aligned}$$

where $n = |N|$. We shall use this result to prove that the invariant approximation property is preserved under taking direct product with a finite group : let H be a discrete group with the IAP and K a finite group. Then the direct product $G = H \times K$ has IAP.

1. INTRODUCTION

The purpose of this paper is to provide an illustration of an interesting and nontrivial interaction between analytic and geometric properties of a group. We provide approximation property of operator algebras associated with discrete

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groups. There are various notions of finite dimensional approximation properties for C^*- algebras and more generally operator algebras. Some of these (approximation properties) notations will be defined in this paper, the reader is referred to [3], [8], [9], [10], [2], [12] and [4] for these a beautiful concept: Haagerup discovery that that the reduced C^*- algebra $\mathbb{F}_n$ has the metric approximation property, Higson and Kasparov's resolution of the Baum-connes conjecture for the Haagerup groups. We studies analytic techniques from operator theory that encapsulate geometric properties of a group. On approximation properties of group C^*- algebras is everywhere; it is powerful, important, backbone of countless breakthroughs.

Roe considered the discrete group of the reduced group C^*- algebra of $C_r^*(G)$ is the fixed point algebra $\{Ad\rho(t) : t \in G\}$ acting on the uniform Roe algebra $C_u^*(G)$ [11]. A discrete group G has natural coarse structure which allows us to define the the uniform Roe algebra, $C_u^*(G)$ [11]. According to [Roe] [11] G has the invariant approximation property (IAP) if

$$C_\lambda^*(G) = C_u^*(G)^G.$$

We give a general exposition of invariant approximation property(IAP), which was initiated by Roe [12]. The main result of paper is the following (see Theorem 1.2). Brodzki, Niblo and Wright [1] show that the uniform Roe algebra of metric space is a coarse invariant up to Mortia equivalence. Next statement can be made a little more precise in the following situation (see Theorem 3.2).

Theorem 1.1. *Let X be a bounded geometry metric space and assume that there is a bijective coarse equivalence*

$$\phi : X \longrightarrow Y \times N,$$

where N is a finite metric space. Then there is an isomorphism

$$\begin{aligned} C_u^*(X) &\cong C_u^*(Y) \otimes C_u^*(N) \\ &\cong C_u^*(Y) \otimes M_n(\mathbb{C}). \end{aligned}$$

where $n = |N|$

We show that the invariant approximation property passes to direct products (see Theorem 1.2).

Theorem 1.2. *Let H be a discrete group with the IAP and K a finite group, then the direct product $G = H \times K$ has IAP.*

In this paper will have a pretty good feel for most aspects of invariant approximation property. In section 2, we recall coarse geometry, uniform Roe algebras. In section 3, we show that a relationship between of uniform Roe algebra attached to coarsely equivalence meric space (see Theorem 3.2). In sections 3 we show that the direct products (see Theorem 3.2).

2. PRELIMINARIES

In this section we shall establish the basic definitions and notations for the category of coarse metric spaces.

Example 1. *[11] Let G be a finitely generated group. Then the bounded coarse structure associated to any word metric on G is generated by the diagonals*

$$\Delta_g = \{(h, hg) : h \in G\}.$$

Let X be a discrete metric space.

Definition 2.1. *[11] We say that discrete metric space X has* bounded geometry *if for all R there exists N in $\mathbb{N}$ such that for all $x \in X$, $|B_R(x)| < N$, where $B(x, r) = \{x \in X : d(y, x) \le r\}$.*

Definition 2.2. *[11] A kernel $\phi : X \times X \longrightarrow \mathbb{C}$,*

- *is* bounded *if there, exists $M > 0$ such that $|\phi(s, t)| < M$ for all $s, t \in X$;*
- *has* finite propagation *if there exists $R > 0$ such that $\phi(s, t) = 0$ if $d(s, t) > R$.*

Let $B(X)$ be a set of bounded finite propagation kernels on $X \times X$. Each such ϕ defines a bounded operator on $\ell^2(X)$ via the usual formula for matrix multiplication

$$\phi * \zeta(s) = \sum_{r \in G} \phi(s, r)\zeta(r) \text{ for } \zeta \in \ell^2(X).$$

Next, we show the operator associated with a bounded kernel is bounded.

Lemma 2.1. *[11] Let X be bounded geometry metric space. An operator associated with a bounded finite propagation kernel is bounded.*

We shall denote the finite propagation kernels on X by $A^\infty(X)$.

Definition 2.3. *[11] The uniform Roe algebra of a metric space X is the closure of $A^\infty(X)$ in the algebra $B(\ell^2(X))$ of bounded operators on X.*

If a discrete group G is equipped with its bounded coarse structure introduced in Example 1 then one can associated with it uniform Roe algebra $C_u^*(G)$ by repeating the above. In this section we will give definition of invariant approximation property. A discrete group G has a natural coarse structure which allows us to define the uniform Roe algebra $C_u^*(G)$. A group G can be equipped with either the left or right-invariant of the metric. A choice of one of the determines whether $C_\lambda^*(G)$ or $C_\rho^*(G)$ is a sublagebra of the uniform Roe algebra $C_u^*(G)$ of G. Hence any element of $\mathbb{C}[G]$ will give use to finite propagation and this assignment extends to an inclusion

$$C_\lambda^*(G) \hookrightarrow C_u^*(G).$$

Next if the metric on G is left-invariant then

$$C_\rho^*(G) \subset C_u^*(G).$$

Let d_1 be the left-invariant metric on G

$$d_1(x,y) = d_1(gx, gy) \ \forall \ g \in G.$$

Let us now choose a right invariant metric for G so that $C_\lambda^*(G) \hookrightarrow C_u^*(G)$. The right regular representation ρ gives use to the adjoint action on $C_u^*(G)$ defined by

$$Ad\rho(g)T = \rho(g)T\rho(g)^* = \rho(g)T\rho(g)^{-1}$$

for all $t \in G$, $T \in C_u^*(G)$. Our remarks above show that elements of $C_\lambda^*(G)$ are invariant with respect to this action and so $C_\lambda^*(G)$ is contained in invariant subalgebra $C_u^*(G)^G$.

Lemma 2.2 ([7]). *If $T \in C_u^*(G)$ has kernel $A(x,y)$, then $Ad\rho(t)T$ has kernel $A(xt, yt)$.*

In general, if $T \in C_u^*(X)$ then $\forall \ x, y \in G$:

$$\begin{aligned} \langle Ad(\rho(t))T\delta_x, \delta_y\rangle &= \left\langle \rho(t)T\rho(t^{-1})\delta_x, \delta_y \right\rangle \\ &= \left\langle T\rho(t^{-1})\delta_x, \rho(t^{-1})\delta_y \right\rangle \\ &= \langle T\delta_{xt}, \delta_{yt}\rangle \,. \end{aligned}$$

So the operator T is $Ad\rho-$ invariant if and only if

$$\forall\ x, y \in X\ \forall\ t \in G\ \ \langle T\delta_{xt}, \delta_{yt}\rangle = \langle T\delta_x, \delta_y\rangle\,.$$

We now define the invariant approximation: property (IAP).

Definition 2.4 ([11]). *We say that G has the* invariant approximation property *(*IAP*) if*

$$C^*_\lambda(G) = C^*_u(G)^G.$$

3. THE IAP PASSES TO DIRECT PRODUCTS WITH FINITE GROUP

In this section, we show that the invariant approximation property is preserved under taking direct product with finite group. We now recall the definition of Morita equivalence:

Definition 3.1 ([1]). *We say that two unital C^*-algebras A and B are Morita equivalent if and only if they are stably isomorphic, which means that $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the algebra of compact operators.*

The following Theorem can be found in [1].

Theorem 3.1 ([1]). *If X and Y are uniformly discrete bounded geometry spaces, and X is coarsely equivalent to Y then, $C^*_u(X)$ is Morita equivalent to $C^*_u(Y)$.*

This statement can be made a little more precise in the following situation.

Theorem 3.2. *Let X be a bounded geometry metric space and assume that there is a bijective coarse equivalence*

$$\phi : X \longrightarrow Y \times N,$$

where Y is a bounded geometry metric space and N is a finite metric space. Then there is an isomorphism

$$\begin{aligned} C^*_u(X) &\cong C^*_u(Y) \otimes C^*_u(N) \\ &\cong C^*_u(Y) \otimes M_n(\mathbb{C}). \end{aligned}$$

where $n = |N|$.

Proof. We shall assume that the bijection ϕ is implemented by means of two maps

$$f : X \longrightarrow Y \text{ and } \pi : X \longrightarrow N$$

so that

$$\phi(x) = (f(x), \pi(x)) \text{ for all } x \in X.$$

The bijection ϕ gives rise to a unitary isomorphism

$$\ell^2(X) \cong \ell^2(Y) \otimes \ell^2(N).$$

This induces a continuous isomorphism

$$\Phi : B(\ell^2(X)) \xrightarrow{\cong} B(\ell^2(Y) \otimes \ell^2(N)) \cong B(\ell^2(Y)) \otimes M_n(\mathbb{C}),$$

Where we use the fact that $\ell^2(N) = \mathbb{C}^n$. We shall show that Φ restricts to an isomorphism

$$\Phi : C_u^*(X) \longrightarrow C_u^*(Y) \otimes M_n(\mathbb{C}).$$

First we need to show that, if T is a finite propagation operator on $\ell^2(X)$ then $\Phi(T) \in C_u^*(Y) \otimes M_n(\mathbb{C})$. For every $i = 1 \ldots n$, let $X_i = \pi^{-1}(i)$ and note that the restriction of f to X_i gives a bijection

$$f|_{X_i} : X_i \xrightarrow{\cong} Y.$$

We shall denote by V_i the corresponding unitary isomorphism

$$V_i : \ell^2(X_i) \xrightarrow{\cong} \ell^2(Y),$$

and let P_i be the projection

$$P_i : \ell^2(X) \longrightarrow \ell^2(X_i).$$

Then any operator $T \in C_u^*(X)$ admits a decomposition

$$T = \sum_{i,j=1}^{n} P_i T P_j,$$

where $P_i T P_j$ is an operator from $\ell^2(X_j)$ to $\ell^2(X_i)$.

Let $S_{i,j} = P_i T P_j$. Then

$$V_i S_{i,j} V_j^* : \ell^2(Y) \longrightarrow \ell^2(Y)$$

is a unitary isomorphism and we have

$$\Phi(S_{i,j}) = V_i S_{i,j} V_j^* \otimes E_{ij},$$

where E_{ij} is the $(i,j)-$th elementary matrix. We want to show that $V_iS_{i,j}V_j^*$ is a finite propagation operator on Y. This fact that $f : X \longrightarrow Y$ is a coarse map. Let $y_1,\ y_2 \in Y$. Then

$$\begin{aligned}\left\langle V_iS_{i,j}V_j^*\delta_{y_1}, \delta_{y_2}\right\rangle &= \left\langle V_iP_iTP_jV_j^*\delta_{y_1}, \delta_{y_2}\right\rangle \\ &= \left\langle TP_jV_j^*\delta_{y_1}, P_iV_i^*\delta_{y_2}\right\rangle \\ &= \left\langle T\delta_{x_1}, \delta_{x_2}\right\rangle,\end{aligned}$$

where x_1 is the preimage of y_1 in X_j and x_2 is the preimage of y_2 in X_i. As T is a bounded propagation operator, there exists $R\ >\ 0$ so that

$$\langle T\delta_{x_1}, \delta_{x_2}\rangle = 0 \text{ when } d(x_1,x_2)\ >\ R.$$

Since f is a coarse map, $\exists\ S\ >\ 0$ such that

$$d_Y(f(x_1), f(x_2))\ >\ S\ \Rightarrow\ d_X(x_1,x_2)\ >\ R.$$

As f is a surjection we now have that for all $y_1,\ y_2 \in Y$ such that $d(y_1,y_2)\ >\ S$, there exist x_1 in X_j, x_2 in X_i such that $d(x_1,x_2)\ >\ R$ and

$$\left\langle V_iS_{i,j}V_j^*\delta_{y_1}, \delta_{y_2}\right\rangle \;=\; \langle T\delta_{x_1}, \delta_{x_2}\rangle = 0.$$

So $V_iS_{i,j}V_j^* \in C_u^*(Y)$ has required. Next, we need to show that Φ is an isomorphism and for this we shall construct an inverse map

$$\Phi^{-1} : C_u^*(Y) \otimes M_n(\mathbb{C}) \longrightarrow C_u^*(X).$$

If $T \otimes E_{ij} \in C_u^*(Y) \otimes M_n(\mathbb{C})$. Then define

$$\Phi^{-1}(T \otimes E_{ij}) = P_iV_i^*TV_jP_j.$$

Using the same argument as before we prove that the operator $P_iV_i^*TV_jP_j$ is of finite propagation, since f is a coarse equivalence. We extend Φ^{-1} by linearity and continuity to a map

$$\Phi^{-1} : C_u^*(Y) \otimes M_n(\mathbb{C}) \longrightarrow C_u^*(X).$$

We need to show that

$$\Phi^{-1}\circ\Phi = \Phi\circ\Phi^{-1} = Id.$$

First we have

$$\begin{aligned}\Phi\circ\Phi^{-1}(T\otimes E_{i,j}) &= \Phi(P_iV_i^*TV_jP_j) \\ &= \sum_{l,k} V_kP_k(P_iV_i^*TV_jP_j)P_lV_l^*\otimes E_{k,l}.\end{aligned}$$

Note that $1 \leq l, k \leq n$

$$P_k P_i = \begin{cases} 0 & \text{if } k \neq i, \\ P_i & \text{if } k = i, \end{cases}$$

and

$$P_j P_l = \begin{cases} 0 & \text{if } l \neq j, \\ P_j & \text{if } l = j. \end{cases}$$

Hence the above sum can be simplified as follows

$$\begin{aligned} \Phi \circ \Phi^{-1}(T \otimes E_{i,j}) &= \sum_{k,l} V_k P_k (P_i V_i^* T V_j P_j) P_l V_l^* \otimes E_{k,l} \\ &= V_i P_i V_i^* T V_j P_j V_j^* \otimes E_{i,j} \, . \end{aligned}$$

Since $P_j|_{\ell^2(X_j)} = id_{X_j}$, we have

$$V_j P_j V_j^* = V_j V_j^* = Id_{X_j},$$

and

$$V_i P_i V_i^* = V_i V_i^* = Id_{X_j},$$

we have

$$\begin{aligned} \Phi \circ \Phi^{-1}(T \otimes E_{i,j}) &= V_i P_i V_i^* T V_j P_j V_j^* \otimes E_{i,j} \\ &= T \otimes E_{i,j} \, . \end{aligned}$$

Moreover:

$$\begin{aligned} \Phi^{-1} \circ \Phi(T) &= \Phi^{-1} \left\{ \sum_{l,k} V_k P_k T P_l V_l^* \otimes E_{k,l} \right\} \\ &= \sum_{k,l} P_i V_i^* V_l P_l T P_k V_k^* V_j P_j \\ &= \sum_{i,j} P_i T P_j \\ &= T. \end{aligned}$$

Therefore

$$\Phi^{-1} \circ \Phi = \Phi \circ \Phi^{-1} = Id.$$

We conclude that

$$C_u^*(X) \cong C_u^*(Y) \otimes M_n(\mathbb{C}) \cong C_u^*(Y) \otimes C_u^*(N).$$

Hence the result follows. □

Next we prove that the invariant approximation property is preserved under taking direct product with finite group.

Theorem 3.3. *Let H be a discrete group with the IAP and K a finite group, then the direct product $G = H \times K$ has IAP.*

Proof. Let us denote the identification $G = H \times K$ by ϕ:

$$\phi : G \xrightarrow{\cong} H \times K.$$

Then

$$C_u^*(G) \cong C_u^*(H \times K).$$

The map ϕ is $G-$ equivariant we have

$$C_u^*(G)^G \cong \left(C_u^*(H \times K)\right)^{H \times K}.$$

By Theorem 3.2, we have,

$$C_u^*(H \times K) \cong C_u^*(H) \otimes C_u^*(K)$$

so that

$$C_u^*(G)^G \cong \left(C_u^*(H \times K)\right)^{H \times K}.$$

Since the identification $G = H \times K$ is a group isomorphism, the unitary isomorphism

$$\ell^2(G) = \ell^2(H) \otimes \ell^2(K)$$

induces a unitary equivalence $\lambda_G \cong \lambda_H \otimes \lambda_K$ and $\rho_G \cong \rho_H \otimes \rho_K$. This means $H \times K$ acts on $C_u^*(H) \otimes C_u^*(K)$ by $Ad\rho_H \otimes Ad\rho_K$ and so

$$C_u^*(H \times K)^{H \times K} \cong C_u^*(H)^H \otimes C_u^*(K)^K.$$

By the same remark,

$$C_\lambda^*(G) \cong C_\lambda^*(H) \otimes C_\lambda^*(K)$$

K is a finite group, so it amenable and so has the IAP, Roe [11]. Since H has the IAP by assumption

$$\begin{aligned} C_u^*(G)^G &= C_u^*(H)^H \otimes C_u^*(K)^K \\ &= C_\lambda^*(H) \otimes C_\lambda^*(K) \\ &= C_\lambda^*(H \times K) \\ &= C_\lambda^*(G). \end{aligned}$$

Therefore

$$C_u^*(G)^G = C_\lambda^*(G).$$

□

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