Application of Layered Medium Green's Function in Low Frequency Models for Numerical Solutions

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Abstract – **An accurate and efficient technique is presented for obtaining numerical solutions of microstrip structures at low frequencies. In this approach, a new form of the electric-field spatial-domain Green's function is used in a symmetrical form that simplifies the discretization of the integral equation using the method of moments (MoM). Hence, a Helmholtz decomposition of the unknown currents is achieved by applying the loop-tree decomposition of the currents. However, the MoM matrix thus obtained still cannot be solved efficiently by iterative solvers due to the large number of iterations required. Consequently, a permutation of the loop-tree currents by a connection matrix is proposed to arrive at a current basis that yields a MoM matrix that can be solved efficiently by iterative solvers.**

I. INTRODUCTION

Scattering solutions using numerical methods have been studied extensively using various types of full wave analysis techniques. However, these techniques are having difficulties because they usually involve the solution of a very large system of linear equations. In this approach, a symmetrical form of electric-field spatial-domain Green's function [1] different from [2] and [3] is applied. Further, the numerical solution of Maxwell's equations at low frequencies is plagued with numerous problems. Because of the discrepant frequency dependence of the solenoidal and irrotational components of the current when the frequency tends to zero, a working numerical method has to include this Helmholtz decomposition and ascribe the requisite frequency dependencies to the solenoidal and irrotational components of the current. This decomposition is achieved by selecting the loop-tree basis [4], [5]. The use of the loop-tree basis, followed by frequency normalization, solves the problem of singular matrices at low frequencies. However, if an iterative solver is used, the iteration count is usually very large and may even diverge for some problems. To overcome this problem, a method of transformation of the matrix equations [6] is also applied. A general type of layered medium microstrip structure as shown in Fig.1 is considered for deriving the layered medium Green's function.

II. FORMULATION

For a geometry of a microstrip structure as shown in Fig. 1, the spectral domain Green's function can be derived in a closed form as the sum of TE and TM to z-waves propagating in the positive and negative z directions. After some manipulations, the spectral domain dyadic Green's function $\tilde{\bar{\mathbf{G}}}$ in the region $z > 0$ can be written in a symmetric form [1] as follows:

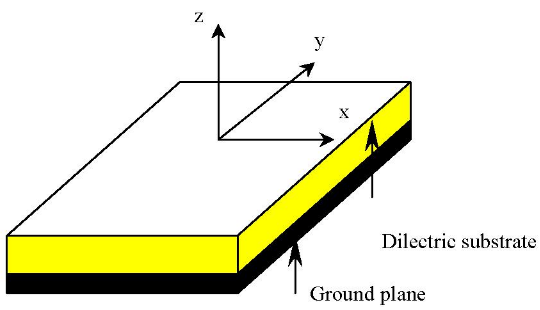

Fig. 1 : Layered medium micrstrip structure

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$$\begin{aligned}\hat{\alpha}\cdot\tilde{\overline{\mathbf{G}}}\cdot\hat{\alpha}' = &\left(\alpha_x\alpha'_x+\alpha_y\alpha'_y\right)\left(\tilde{g}^P+\tilde{g}^{\mathrm{TE},R}\right)\\ &+\alpha_z\alpha'_z\left(\tilde{g}^P+\tilde{g}^{\mathrm{TM},R}\right)\\ &+\frac{1}{k^2}\hat{\alpha}\cdot\nabla\nabla\cdot\hat{\alpha}'\,\tilde{g}^P\\ &+\frac{1}{k^2}\hat{\alpha}\cdot\nabla\nabla\cdot\hat{\alpha}'_I\tilde{g}^{\mathrm{TM},R}\\ &-\hat{\alpha}\cdot\nabla_s\nabla_s\cdot\hat{\alpha}'_I\tilde{g}^{\mathrm{EM}}\end{aligned}\qquad(1)$$

where $\hat{\alpha}=\alpha_x\hat{x}+\alpha_y\hat{y}+\alpha_z\hat{z}$,

$\hat{\alpha}'=\alpha'_x\hat{x}+\alpha'_y\hat{y}+\alpha'_z\hat{z}$,

$\hat{\alpha}'=-\alpha'_x\hat{x}-\alpha'_y\hat{y}+\alpha'_z\hat{z}$,

$$\tilde{g}^P=-\frac{\omega\mu_0}{8\pi^2}\frac{e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)}}{k_z}e^{ik_z|z-z'|},$$

$$\tilde{g}^{\mathrm{TM},R}=-\frac{\omega\mu_0}{8\pi^2}\tilde{R}^{\mathrm{TM}}\frac{e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)}}{k_z}e^{ik_z(z+z'+2h)},$$

$$\tilde{g}^{\mathrm{TE},R}=-\frac{\omega\mu_0}{8\pi^2}\tilde{R}^{\mathrm{TE}}\frac{e^{i\mathbf{k}_s\cdot(\mathbf{r}_s-\mathbf{r}'_s)}}{k_z}e^{ik_z(z+z'+2h)},$$

$\tilde{g}^{\mathrm{EM}}=\left(\tilde{g}^{\mathrm{TM},R}+\tilde{g}^{\mathrm{TE},R}\right)/k_s^2$ and

k is the wavenumber in free space, $k_s^2=k_x^2+k_y^2$ and $\tilde{R}^{\mathrm{TM,TE}}$ is the generalized reflection coefficient of the layered medium. The spectral integration of (1) yields the spatial domain Green's function as follows:

$$\begin{aligned}\hat{\alpha}\cdot\overline{\mathbf{G}}\cdot\hat{\alpha}' = &\left(\alpha_x\alpha'_x+\alpha_y\alpha'_y\right)\left(g^P+g^{\mathrm{TE},R}\right)\\ &+\alpha_z\alpha'_z\left(g^P+g^{\mathrm{TM},R}\right)\\ &+\frac{1}{k^2}\hat{\alpha}\cdot\nabla\nabla\cdot\hat{\alpha}'\,g^P\\ &+\frac{1}{k^2}\hat{\alpha}\cdot\nabla\nabla\cdot\hat{\alpha}'_I g^{\mathrm{TM},R}\\ &-\hat{\alpha}\cdot\nabla_s\nabla_s\cdot\hat{\alpha}'_I g^{\mathrm{EM}}\end{aligned}\qquad(2)$$

where $g^\beta=\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\tilde{g}^\beta dk_x dk_y$ and

$\beta=P,(\mathrm{TE},R),(\mathrm{TM},R)$ and EM

Using the dyadic electric-field Green's function $\overline{\mathbf{G}}$ for the layered medium, an electric-field integral equation (EFIE) can be constructed by enforcing the total electric-field tangential to the surface S to vanish

$$\begin{aligned}&\hat{z}\times\int_S\left(g^P(\mathbf{r},\mathbf{r}')+g^{\mathrm{TE},R}(\mathbf{r},\mathbf{r}')\right)\mathbf{J}(\mathbf{r}')d\mathbf{r}'\\ &+\hat{z}\times\nabla_s\int_S\frac{1}{k^2}g^P(\mathbf{r},\mathbf{r}')\nabla'_s\cdot\mathbf{J}(\mathbf{r}')d\mathbf{r}'\\ &-\hat{z}\times\nabla_s\int_S\frac{1}{k^2}g^{\mathrm{TM},R}(\mathbf{r},\mathbf{r}')\nabla'_s\cdot\mathbf{J}(\mathbf{r}')d\mathbf{r}'\\ &+\hat{z}\times\nabla_s\int_S g^{\mathrm{EM}}\nabla'_s\cdot\mathbf{J}(\mathbf{r}')d\mathbf{r}'\\ &=-\hat{z}\times\mathbf{E}^{\mathrm{inc}}(\mathbf{r})\end{aligned}\qquad(3)$$

where $\mathbf{E}^{\mathrm{inc}}(\mathbf{r})$ can be the field of a impinging plane wave or the field created by a finite source residing within the microstrip structure. Using the loop-tree basis function designed for low-frequency problems,

$$\mathbf{J}(\mathbf{r}')=\sum_{n=1}^{N_L}I_{Ln}\mathbf{J}_{Ln}(\mathbf{r}')+\sum_{n=1}^{N_T}I_{Tn}\mathbf{J}_{Tn}(\mathbf{r}')\qquad(4)$$

we discretize the EFIE into a linear algebraic system of equations, where $\mathbf{J}_{Ln}(\mathbf{r}')$ and $\mathbf{J}_{Tn}(\mathbf{r}')$ are the divergence-free surface-loop basis and the nondivergence-free surface-tree basis, respectively. By substituting (4) into (3), testing with $\mathbf{J}_{Lm}(\mathbf{r})$ and $\mathbf{J}_{Tm}(\mathbf{r})$, and applying $\nabla_s\cdot\mathbf{J}_{Lm}(\mathbf{r})=0$ and $\nabla_s\cdot\mathbf{J}_{Ln}(\mathbf{r}')=0$, we simplify to a matrix form as follows:

$$\begin{bmatrix}\mathbf{Z}_{LL} & \mathbf{Z}_{LT}\\ \mathbf{Z}_{TL} & \mathbf{Z}_{TT}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{I}_L\\ \mathbf{I}_T\end{bmatrix}=\begin{bmatrix}\mathbf{V}_L\\ \mathbf{V}_T\end{bmatrix}\qquad(5)$$

where $\mathbf{V}_L=-\left\langle\mathbf{J}_L(\mathbf{r}),\mathbf{E}^{\mathrm{inc}}(\mathbf{r})\right\rangle$,

$\mathbf{V}_T=-\left\langle\mathbf{J}_T(\mathbf{r}),\mathbf{E}^{\mathrm{inc}}(\mathbf{r})\right\rangle$,

$\overline{\mathbf{Z}}_{LL}=\left\langle\mathbf{J}_L(\mathbf{r}),g^V(\mathbf{r},\mathbf{r}'),\mathbf{J}^t_L(\mathbf{r}')\right\rangle$,

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$$\overline{\mathbf{Z}}_{LT} = \left\langle \mathbf{J}_L(\mathbf{r}), g^V(\mathbf{r},\mathbf{r}'), \mathbf{J}_T^t(\mathbf{r}') \right\rangle,$$

$$\overline{\mathbf{Z}}_{TL} = \left\langle \mathbf{J}_T(\mathbf{r}), g^V(\mathbf{r},\mathbf{r}'), \mathbf{J}_L^t(\mathbf{r}') \right\rangle = \overline{\mathbf{Z}}_{LT}^t,$$

$$\overline{\mathbf{Z}}_{TT} = \left\langle \mathbf{J}_T(\mathbf{r}), g^V(\mathbf{r},\mathbf{r}'), \mathbf{J}_T^t(\mathbf{r}') \right\rangle - \frac{1}{k^2}\left\langle \nabla_s \cdot \mathbf{J}_T(\mathbf{r}), g^S(\mathbf{r},\mathbf{r}'), \nabla_s' \cdot \mathbf{J}_T^t(\mathbf{r}') \right\rangle$$

$g^V(\mathbf{r},\mathbf{r}') = g^P(\mathbf{r},\mathbf{r}') + g^{TE,R}(\mathbf{r},\mathbf{r}')$,
$g^S(\mathbf{r},\mathbf{r}') = \left(g^P(\mathbf{r},\mathbf{r}') - g^{\mathrm{TM},R}(\mathbf{r},\mathbf{r}') + k^2 g^{\mathrm{EM}}\right)$ and

$$\left\langle \mathbf{A}(\mathbf{r}), g(\mathbf{r},\mathbf{r}'), \mathbf{B}^t(\mathbf{r}') \right\rangle = \int \mathbf{A}(\mathbf{r})d\mathbf{r} \cdot \int g(\mathbf{r},\mathbf{r}')\mathbf{B}^t(\mathbf{r}')d\mathbf{r}'.$$

Also, $\mathbf{J}_L(\mathbf{r}')$, $\mathbf{J}_L(\mathbf{r})$, $\mathbf{I}_L$, $\mathbf{J}_T(\mathbf{r}')$, $\mathbf{J}_T(\mathbf{r})$, and $\mathbf{I}_T$ are column vectors containing $\mathbf{J}_{Ln}(\mathbf{r}')$, $\mathbf{J}_{Lm}(\mathbf{r})$, I_{Ln}, $\mathbf{J}_{Tn}(\mathbf{r}')$, $\mathbf{J}_{Tm}(\mathbf{r})$, and I_{Tn}, respectively.

When the frequency $\omega \to 0$ $(k \to 0)$, the matrix equation is unbalanced and ill conditioned. Since the lower-right block of the matrix becomes dominant in an electric field equation, frequency normalization can be used to balance the matrix as below:

$$\begin{bmatrix} \overline{\mathbf{Z}}_{LL}(O(1)) & k\overline{\mathbf{Z}}_{LT}(O(\omega)) \\ k\overline{\mathbf{Z}}_{TL}(O(\omega)) & k^2\overline{\mathbf{Z}}_{TT}(O(1)) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I}_L(O(1)) \\ \frac{1}{k}\mathbf{I}_T(O(1)) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_L(O(\omega)) \\ k\mathbf{V}_T(O(\omega)) \end{bmatrix} \quad (6)$$

The above matrix equation is balanced and can be solved by the direct inversion method. However, if the matrix equation is solved by iterative solvers, the iteration count is usually very large. Even though the electrostatic part converges very slowly, the magnetostatic part converges rapidly. However, the electrostatic problem based on pulse basis converges rapidly. Therefore, the charge basis arising from the divergence of the current basis is the main reason for the matrix ill conditioning. To avoid this, we transform the electrostatic part in the matrix equation by basis rearrangement so that the resultant matrix reduces to that based on the pulse basis in the static limit as given in [6].

Expanding the surface charge densities in terms of pulse basis $\rho(\mathbf{r}) = \sum_{n=1}^{N_P} Q_n P_n(\mathbf{r})$ and applying the condition for the charge neutral system, we obtain the expression for the surface charge density as follows:

$$\rho(\mathbf{r}) = \sum_{n=1}^{N_P-1}\left[P_n(\mathbf{r}) - C_{nN_P}P_{N_P}(\mathbf{r})\right]Q_n = \mathbf{N}^t(\mathbf{r})\cdot\mathbf{Q} \quad (7)$$

where $C_{nN_P} = \int_{S_n} P_n(\mathbf{r})dS \left[\int_{S_{N_P}} P_{N_P}(\mathbf{r})dS\right]^{-1}$,

$\mathbf{N}(\mathbf{r})$ and $\mathbf{Q}$ are vectors of length $N_P - 1$. Using the current continuity condition $\nabla \cdot \mathbf{J}(\mathbf{r}) = i\omega\rho(\mathbf{r})$, applying $\nabla_s \cdot \mathbf{J}_L(\mathbf{r}) = 0$ and taking the inner product with $\mathbf{P}(\mathbf{r})$, we have

$$\left\langle \mathbf{P}(\mathbf{r}), \nabla \cdot \mathbf{J}_T^t(\mathbf{r}) \right\rangle \cdot \mathbf{I}_T = i\omega \left\langle \mathbf{P}(\mathbf{r}), \mathbf{N}^t(\mathbf{r}) \right\rangle \cdot \mathbf{Q} \quad (8)$$

In this manner, $\left\langle \mathbf{P}(\mathbf{r}), \mathbf{N}^t(\mathbf{r}) \right\rangle$ is a diagonal matrix and we can rewrite (8) as

$$\overline{\mathbf{K}} \cdot \mathbf{I}_T = i\omega\, \mathbf{Q} \quad (9)$$

where $\overline{\mathbf{K}}$ is a square matrix. Applying (8) in (6), we obtain the transformed matrix as below with good spectral property and the matrix equation converges rapidly.

$$\begin{bmatrix} -i\overline{\mathbf{Z}}_{LL} & \omega\overline{\mathbf{Z}}_{LT}\cdot\overline{\mathbf{K}}^{-1} \\ k^2/\omega\, \overline{\mathbf{Z}}_{TL} & ik^2\overline{\mathbf{Z}}_{TT}\cdot\overline{\mathbf{K}}^{-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I}_L \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} -i\mathbf{V}_L \\ k^2/\omega\, \mathbf{V}_T \end{bmatrix} \quad (10)$$

Also a symmetrical form of impedance matrix can be obtained from (10) as below:

$$\begin{bmatrix} \overline{\mathbf{Z}}_{LL} & k\overline{\mathbf{Z}}_{LT}\cdot\overline{\mathbf{K}}^{-1} \\ k\,\overline{\mathbf{K}}^{t^{-1}}\cdot\overline{\mathbf{Z}}_{TL} & k^2\overline{\mathbf{K}}^{t^{-1}}\cdot\overline{\mathbf{Z}}_{TT}\cdot\overline{\mathbf{K}}^{-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{I}_L \\ \frac{i\omega}{k}\mathbf{Q} \end{bmatrix} = \begin{bmatrix} -i\mathbf{V}_L \\ k\,\overline{\mathbf{K}}^{t^{-1}}\cdot\mathbf{V}_T \end{bmatrix} \quad (11)$$

III. NUMERICAL RESULTS

Current distribution patterns have been obtained for a θ-

polarized plane wave impinging the rectangular surface with the incident angle of $\theta=60^0$ & $\phi=0^0$ and a current source excitation at the center of the surface at the low frequencies (500 - 1000 Hz) as shown in Fig.2a and Fig.2b, respectively. The current coefficients have been obtained with less than 100 iterations for the unknowns in the range of 10^4. Also a five-fold reduction of the iteration count has been observed after the basis rearrangement is executed.

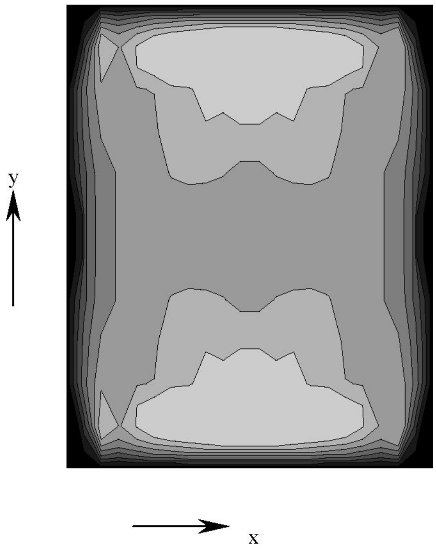

Fig.2a: Current distribution pattern when a θ-polarized plane wave impinging the surface with an incident angle of $\theta=60^0$ and $\phi=0^0$.

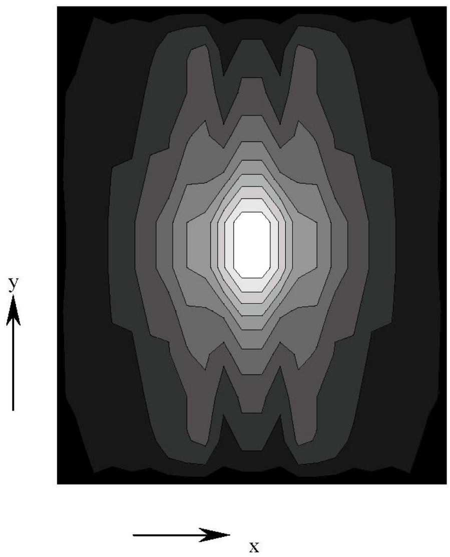

Fig.2b: Current distribution pattern when the excitation is given by a current source at the center.

IV. CONCLUSIONS

A symmetric form of layered medium Green's function is successfully used to analyze the microstrip structure at low frequencies with the help of loop-tree basis functions, frequency normalization as well as the basis rearrangement. It converges fast and no low-frequency breakdown occurs in the numerical computations. It is also noted that the current distribution patterns shown in Fig.2a and Fig.2b satisfy the symmetrical properties that ensure the validity of the simulation results. The present formulation could be extended to multi-layered medium structures by deriving the respective special domain Green's functions. Since the iterative method is successfully implemented for the low frequency problems, this method would be capable of solving large-scale low frequency problems by cautious integration of efficient algorithms such as Multi Level Fast Multipole Algorithm (MLFMA).

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