VINGNANAM Research Conference

21st of July 2022

VRC–2022

Faculty of Science
University of Jaffna
Sri Lanka

Jointly Organized by

Faculty of Science
University of Jaffna
Sri Lanka

Western Norway
University of
Applied Sciences

Nested radicals and coinduction principle

J. K. Ratnayake[1]*, M. Annanthakrishna[2] and J. N. Senadheera[1]

[1] *Department of Mathematics, Faculty of Applied Science, University of Sri Jayewardenepura, Sri Lanka*

[2] *Department of Mathematics and Statistics, Faculty of Science, University of Jaffna, Sri Lanka*

** Correspondence: jratnayake@sjp.ac.lk*

Infinite nested radicals of the form $\pm\sqrt{2 \pm \sqrt{2 \pm \sqrt{2 \pm \cdots}}}$ arise from the iterated function system $\pm\sqrt{2 \pm x}$. These functions can be combined to construct a coalgebra $\alpha\colon [-2,2] \to F([-2,2])$ for an endofunctor F on the category of bi-pointed sets and metric spaces, which takes two copies of the given set and glues it along a point. The final F-coalgebra for this functor is $e\colon [-1,1] \to F([-1,1])$, where the coalgebra structure is related to the iterated function system $0.5x \pm 0.5$. The coinduction principle now implies that there is a unique map $g\colon [-2,2] \to [-1,1]$ satisfying the property $F(g) \circ \alpha = e \circ g$. In this paper, we use this formulation to derive the relationship between nested radicals and numbers in the unit interval. For example, we conclude that every finite nested radical of the given form is associated with a dyadic rational in the interval $[-1,1]$ via the function g and every number in the interval $[-2,2]$ has an infinite nested radical expansion of the given type. The results we present are well known and the proofs given here bear some similarity to that are based on conjugate dynamical systems. However, the proof method we present gives a new insight to how category theoretic notions such as the theory of coalgebras and coinduction principle can be relevant in formulating and understanding such problems. In particular, we believe that these methods are applicable to a more general context. For example, one can speculate that there are similar implications to the dynamics of $x^2 - c$ for $c \geq 2$, where $c = 2$ is what we have considered here.

Keywords: *Coinduction, Co-algebra, Nested radicals, Iterated function systems.*