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S-Spectrum and the quaternionic Cayley transform of an operator

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ABSTRACT

In this paper we define the quaternionic Cayley transformation of a densely defined, symmetric, quaternionic right linear operator and formulate a general theory of defect number in a right quaternionic Hilbert space. This study investigates the relation between the defect number and S-spectrum, and the properties of the Cayley transform in the quaternionic setting.

1. Introduction

Self-adjoint operators play an important role in the Dirac–von Neumann formulation of quantum mechanics. In complex and in quaternionic quantum mechanics states are described by vectors of a separable complex (resp. quaternionic) Hilbert space and the observables are represented by self-adjoint operators on the respective Hilbert space. By Stone's theorem on one parameter unitary groups, self-adjoint operators are the infinitesimal generators of unitary groups of time evolution.

The self-adjointness in a Hilbert space is stronger than being symmetric. Even though the difference is a technical issue, it is very important. For example, the spectral theorem only applies to self-adjoint operators but not to symmetric operators. In this regard, the following question arises in several contexts: if an operator A on a Hilbert space is symmetric, when does it have self-adjoint extensions? In the complex case, an answer is provided by the Cayley transform of a self-adjoint operator and the deficiency indices.

Due to the non-commutativity, in the quaternionic case there are three types of Hilbert spaces: left, right, and two-sided, depending on how vectors are multiplied by scalars. This fact can entail several problems. For example, when a Hilbert space $\mathcal{H}$ is one-sided (either left or right) the set of linear operators acting on it does not have a linear structure. Moreover, in a one sided quaternionic Hilbert space, given a linear operator T and a quaternion $\mathfrak{q} \in \mathbb{H}$, in general we have that $(\mathfrak{q}T)^\dagger \neq \overline{\mathfrak{q}}T^\dagger$

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(see [1] for details). These restrictions can severely prevent the generalization to the quaternionic case of results valid in the complex setting. Even though most of the linear spaces are one-sided, it is possible to introduce a notion of multiplication on both sides by fixing an arbitrary Hilbert basis of $\mathcal{H}$. This fact allows to have a linear structure on the set of linear operators, which is a minimal requirement to develop a full theory. Thus, the framework of this paper is a right quaternionic Hilbert space equipped with a left multiplication, introduced by fixing a Hilbert basis. As in the complex case, one may introduce a suitable notion of Cayley type transform of symmetric linear operators. The idea of considering Cayley transform of linear operators is due to von Neumann [2] who formally replaced the variable in a Cayley transform by a symmetric operator. The idea was further extended to other types of linear operators but always with the purpose of getting information of the given operator by studying the properties of its Cayley transform. A quaternionic Cayley transform of linear operators appeared in [3,4]; however, the type of transform and the underlying notion of spectrum differ from the one treated in this paper.

In this paper, we define the Cayley transform of densely defined symmetric operators satisfying suitable assumptions. We will prove that this notion of Cayley transform possesses several properties, in particular it is an isometry and allows to prove a characterization of self-adjointness.

The plan of the paper is the following. The paper consists of four sections, besides the Introduction. In Section 2 we collect some preliminary notations and results on quaternions, quaternionic Hilbert spaces and Hilbert bases. In Section 3 we introduce right linear operators and some of their properties, the left multiplication, we introduce the notion of deficiency subspace and defect number of an operator at a point also proving some new results in this framework. In Section 4 we study the deficiency indices of isometric operators and we define the notion of quaternionic Cayley transform for a linear symmetric operator (satisfying suitable hypotheses) and study its main properties. In particular, we show that a linear operator is self-adjoint if and only if its Cayley transform is unitary. In the fifth and last section, we show that the Cayley transform that we have defined based on the choice of a Hilbert basis, in order to have a left multiplication and thus a two-sided Hilbert space, in fact does not depend on this choice.

2. Mathematical preliminaries

In order to make the paper self-contained, we recall some facts about quaternions which may not be well-known. For details we refer the reader to [5–7].

2.1. Quaternions

Let $\mathbb{H}$ denote the field of all quaternions and $\mathbb{H}^*$ the group (under quaternionic multiplication) of all invertible quaternions. A general quaternion can be written as

$$\mathrm{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}; \qquad q_0; q_1; q_2; q_3 \in \mathbb{R};$$

where $\mathbf{i}; \mathbf{j}; \mathbf{k}$ are the three quaternionic imaginary units, satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}; \ \mathbf{jk} = \mathbf{i} = -\mathbf{kj}; \ \mathbf{ki} = \mathbf{j} = -\mathbf{ik}$. The quaternionic conjugate of q is

$$\overline{\mathrm{q}} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3;$$

while $|\mathrm{q}| = (\mathrm{q}\overline{\mathrm{q}})^{1=2}$ denotes the usual norm of the quaternion q. If q is non-zero element, it has inverse $\mathrm{q}^{-1} = \frac{\overline{\mathrm{q}}}{|\mathrm{q}|^2}$: Finally, the set

$$\mathbb{S} = \{I = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mid x_1; x_2; x_3 \in \mathbb{R}; \ x_1^2 + x_2^2 + x_3^2 = 1\};$$

contains all the elements whose square is -1. It is a 2-dimensional sphere in $\mathbb{H}$ identified with $\mathbb{R}^4$.

2.2. Quaternionic Hilbert spaces

In this subsection we discuss right quaternionic Hilbert spaces. For more details we refer the reader to [5–7].

2.2.1. Right quaternionic Hilbert Space

Let $V_{\mathbb{H}}^R$ be a vector space under right multiplication by quaternions. For ; ; ! $\in V_{\mathbb{H}}^R$ and $\mathrm{q} \in \mathbb{H}$, the inner product

$$\langle \cdot \mid \cdot \rangle : V_{\mathbb{H}}^R \times V_{\mathbb{H}}^R \longrightarrow \mathbb{H}$$

satisfies the following properties:

(i) $\overline{\langle \ \mid \ \rangle} = \langle \ \mid \ \rangle$
(ii) $\| \ \|^2 = \langle \ \mid \ \rangle > 0$ unless $= 0$, a real norm
(iii) $\langle \ \mid \ + ! \rangle = \langle \ \mid \ \rangle + \langle \ \mid ! \rangle$
(iv) $\langle \ \mid \ \mathrm{q}\rangle = \langle \ \mid \ \rangle \mathrm{q}$
(v) $\langle \ \mathrm{q} \mid \ \rangle = \overline{\mathrm{q}}\langle \ \mid \ \rangle$

where $\overline{\mathfrak{q}}$ stands for the quaternionic conjugate. It is always assumed that the space $V_{\mathbb{H}}^{R}$ is complete under the norm given above and separable. Then, together with $\langle\cdot\mid\cdot\rangle$ this defines a right quaternionic Hilbert space. Quaternionic Hilbert spaces share many of the standard properties of complex Hilbert spaces.

The next two propositions can be established following the proof of their complex counterparts, see e.g. [6,7].

Proposition 2.1. *Let $\mathcal{O}=\{\varphi_k \mid k\in N\}$ be an orthonormal subset of $V_{\mathbb{H}}^{R}$, where N is a countable index set. Then following conditions are pairwise equivalent:*

(a) *The closure of the linear combinations of elements in $\mathcal{O}$ with coefficients on the right is $V_{\mathbb{H}}^{R}$.*
(b) *For every $\phi,\psi\in V_{\mathbb{H}}^{R}$, the series $\sum_{k\in N}\langle\phi\mid\varphi_k\rangle\langle\varphi_k\mid\psi\rangle$ converges absolutely and it holds:*

$$\langle\phi\mid\psi\rangle=\sum_{k\in N}\langle\phi\mid\varphi_k\rangle\langle\varphi_k\mid\psi\rangle.$$

(c) *For every $\phi\in V_{\mathbb{H}}^{R}$, it holds:*

$$\|\phi\|^2=\sum_{k\in N}|\langle\varphi_k\mid\phi\rangle|^2.$$

(d) $\mathcal{O}^{\perp}=\{0\}$.

Definition 2.2. The set $\mathcal{O}$ as in Proposition 2.1 is called a *Hilbert basis* of $V_{\mathbb{H}}^{R}$.

Proposition 2.3. *Every quaternionic Hilbert space $V_{\mathbb{H}}^{R}$ has a Hilbert basis. All the Hilbert bases of $V_{\mathbb{H}}^{R}$ have the same cardinality. Furthermore, if $\mathcal{O}$ is a Hilbert basis of $V_{\mathbb{H}}^{R}$, then every $\phi\in V_{\mathbb{H}}^{R}$ can be uniquely decomposed as follows:*

$$\phi=\sum_{k\in N}\varphi_k\langle\varphi_k\mid\phi\rangle,$$

where the series $\sum_{k\in N}\varphi_k\langle\varphi_k\mid\phi\rangle$ converges absolutely in $V_{\mathbb{H}}^{R}$.

It should be noted that once a Hilbert basis is fixed, every left (resp. right) quaternionic Hilbert space also becomes a right (resp. left) quaternionic Hilbert space [6,7]. See next Section 3.2 for more details.

The field of quaternions $\mathbb{H}$ itself can be turned into a left quaternionic Hilbert space by defining the inner product $\langle\mathfrak{q}\mid\mathfrak{q}'\rangle=\mathfrak{q}\overline{\mathfrak{q}'}$ or into a right quaternionic Hilbert space with $\langle\mathfrak{q}\mid\mathfrak{q}'\rangle=\overline{\mathfrak{q}}\mathfrak{q}'$.

3. Right quaternionic linear operators and some basic properties

In this section we shall define right $\mathbb{H}$-linear operators and recall some basis properties. Most of them are very well known. In this manuscript, we follow the notations in [8,6]. We shall also prove some results pertinent to the development of the paper. To the best of our knowledge the results we prove in Sections 3.3 and 3.4 do not appear in the literature.

Definition 3.1. A mapping $A:\mathcal{D}(A)\subseteq V_{\mathbb{H}}^{R}\longrightarrow V_{\mathbb{H}}^{R}$, where $\mathcal{D}(A)$ stands for the domain of A, is said to be right $\mathbb{H}$-linear operator or, for simplicity, right linear operator, if

$$A(\phi\mathfrak{a}+\psi\mathfrak{b})=(A\phi)\mathfrak{a}+(A\psi)\mathfrak{b},\quad\text{if}\quad\phi,\psi\in\mathcal{D}(A)\quad\text{and}\quad\mathfrak{a},\mathfrak{b}\in\mathbb{H}.$$

The set of all right linear operators will be denoted by $\mathcal{L}(V_{\mathbb{H}}^{R})$ and the identity linear operator on $V_{\mathbb{H}}^{R}$ will be denoted by $\mathbb{I}_{V_{\mathbb{H}}^{R}}$. For a given $A\in\mathcal{L}(V_{\mathbb{H}}^{R})$, the range and the kernel will be

$$\text{ran}(A)=\{\psi\in V_{\mathbb{H}}^{R}\mid A\phi=\psi\quad\text{for}\quad\phi\in\mathcal{D}(A)\}$$
$$\ker(A)=\{\phi\in\mathcal{D}(A)\mid A\phi=0\}.$$

We call an operator $A\in\mathcal{L}(V_{\mathbb{H}}^{R})$ bounded if

$$\|A\|=\sup_{\|\phi\|=1}\|A\phi\|<\infty,$$

or equivalently, there exist $K\geq 0$ such that $\|A\phi\|\leq K\|\phi\|$ for all $\phi\in\mathcal{D}(A)$. The set of all bounded right linear operators will be denoted by $\mathcal{B}(V_{\mathbb{H}}^{R})$.

Assume that $V_{\mathbb{H}}^{R}$ is a right quaternionic Hilbert space, A is a right linear operator acting on it. Then, there exists a unique linear operator $A^{\dagger}$ such that

$$\langle\phi\mid A\psi\rangle=\langle A^{\dagger}\phi\mid\psi\rangle;\quad\text{for all}\quad\psi\in\mathcal{D}(A),\ \phi\in\mathcal{D}(A^{\dagger}),\tag{3.1}$$

where the domain $\mathcal{D}(A^{\dagger})$ of $A^{\dagger}$ is defined by

$$\mathcal{D}(A^{\dagger})=\{\phi\in V_{\mathbb{H}}^{R}\mid\exists\varphi\text{ such that }\langle\phi\mid A\psi\rangle=\langle\varphi\mid\psi\rangle\}.$$

3.1. Symmetry and self-adjointness

Let $A : \mathcal{D}(A) \longrightarrow V_{\mathbb{H}}^{R}$ and $B : \mathcal{D}(B) \longrightarrow V_{\mathbb{H}}^{R}$ be quaternionic linear operators. As usual, we write $A \subset B$ if $D(A) \subset \mathcal{D}(B)$ and $B|_{\mathcal{D}(A)} = A$. In this case, B is said to be an extension of A.

Definition 3.2. A right linear operator $A : \mathcal{D}(A) \longrightarrow V_{\mathbb{H}}^{R}$ with dense domain is said to be

(a) *symmetric*, if $A \subset A^{\dagger}$.
(b) *anti-symmetric*, if $A \subset -A^{\dagger}$.
(c) *self-adjoint*, if $A = A^{\dagger}$.
(d) *unitary*, if $\mathcal{D}(A) = V_{\mathbb{H}}^{R}$ and $AA^{\dagger} = A^{\dagger}A = \mathbb{I}_{V_{\mathbb{H}}^{R}}$.
(e) *closed*, if the graph $\mathcal{G} := \mathcal{D}(A) \oplus \text{ran}(A)$ of A is closed in $V_{\mathbb{H}}^{R} \times V_{\mathbb{H}}^{R}$, equipped with the product topology.
(f) *closable*, if it admits closed operator extensions. In this case, the *closure* $\overline{A}$ of A is the smallest closed extension and its domain and action are

- $\mathcal{D}(\overline{A}) := \{\ \ \in V_{\mathbb{H}}^{R} \mid \exists \ \ \in V_{\mathbb{H}}^{R} \text{ s.t. } \forall \{\ _n\} \subset \mathcal{D}(A) \text{ with } \ _n \to \ ;\ A\ _n \to \ \}$
- $\overline{A}\ \ = \ .$

Proposition 3.3. *A right linear operator $A : \mathcal{D}(A) \longrightarrow V_{\mathbb{H}}^{R}$ is closed if and only if for any sequence $\{\ _n\}$ in $\mathcal{D}(A)$ such that $\ _n \longrightarrow$ with $A\ _n = \ _n \longrightarrow$ in $V_{\mathbb{H}}^{R}$, then $= A\ $.*

Proof. It is straightforward from the definition of closed operators.

Proposition 3.4. *Let $A : \mathcal{D}(A) \longrightarrow V_{\mathbb{H}}^{R}$ be densely defined right linear operator. Then*

(a) *$A^{\dagger}$ is closed.*
(b) *A is closable if and only if $\mathcal{D}(A^{\dagger})$ is dense in $V_{\mathbb{H}}^{R}$, and $\overline{A} = A^{\dagger\dagger}$.*
(c) *$ran(A)^{\perp} = \ker(A^{\dagger})$ and $\ker(A) \subset ran(A^{\dagger})^{\perp}$.*
Furthermore, if $\mathcal{D}(A^{\dagger})$ is dense in $V_{\mathbb{H}}^{R}$ and A is closed, then $\ker(A) = ran(A^{\dagger})^{\perp}$.

Proposition 3.5. *Let $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^{R} \longrightarrow V_{\mathbb{H}}^{R}$ be a right linear operator. If A is closed and satisfies the condition that there exists $C > 0$ such that*

$$\|A\ \ \| \geq C\|\ \ \|; \tag{3.2}$$

for all $\ \in D(A)$, then $ran(A)$ is closed.

Proof. Let $\ \in \overline{\text{ran}(A)}$, then there exists a sequence $\{\ _n\}$ in $\mathcal{D}(A)$ such that $A\ _n \longrightarrow$. Then by (3.2), we know that $\{\ _n\}$ is a Cauchy sequence in $V_{\mathbb{H}}^{R}$ as $\{A\ _n\}$ is Cauchy. Therefore $\ _n \longrightarrow$ for some $\ \in V_{\mathbb{H}}^{R}$. From the Proposition 3.3, we have $A\ \ = \ $. This completes the proof.

Proposition 3.6. *The right linear operator $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^{R} \longrightarrow V_{\mathbb{H}}^{R}$ is symmetric if and only if $\langle A\ \ \mid\ \ \rangle \in \mathbb{R}$, for all $\ \in \mathcal{D}(A)$.*

Proof. If A is symmetric, then the statement has been proved in [6], Proposition 2.17 (b), but since the proof is short, we repeat it for completeness: for any $\ \in \mathcal{D}(A)$, $\langle A\ \ \mid\ \ \rangle = \langle\ \ \mid A\ \ \rangle = \overline{\langle A\ \ \mid\ \ \rangle}$. That is, $\langle A\ \ \mid\ \ \rangle \in \mathbb{R}$, for all $\ \in \mathcal{D}(A)$. To show the converse, suppose that $\langle A\ \ \mid\ \ \rangle \in \mathbb{R}$, for all $\ \in \mathcal{D}(A)$. The polarization identity (see, for example, [6]) is given by the formula

$$\langle\ \ \mid\ \ \rangle = \frac{1}{4}(\|\ \ +\ \ \| - \|\ \ -\ \ \|) + \frac{1}{4}\sum_{=\mathbf{i};\mathbf{j};\mathbf{k}}(\|\ \ +\ \ \| - \|\ \ -\ \ \|)\ ; \tag{3.3}$$

where $\ ;\ \ \in V_{\mathbb{H}}^{R}$. Now using this identity, we can immediately see that

$$\langle A\ \ \mid\ \ \rangle = \langle\ \ \mid A\ \ \rangle;\ \text{for all}\ \ \in \mathcal{D}(A);$$

that is, A is symmetric. Hence the result follows.

3.2. Left scalar multiplications on $V_{\mathbb{H}}^{R}$

We shall extract the definition and some properties of left scalar multiples of vectors on $V_{\mathbb{H}}^{R}$ from [6] as needed for the development of the manuscript. The left scalar multiple of vectors on a right quaternionic Hilbert space is an extremely non-canonical operation associated with a choice of preferred Hilbert basis. From Proposition 2.3, $V_{\mathbb{H}}^{R}$ has a Hilbert basis

$$\mathcal{O} = \{'\ _k \mid k \in N\}; \tag{3.4}$$

where N is a countable index set. The left scalar multiplication on $V_{\mathbb{H}}^R$ induced by $\mathcal{O}$ is defined as the map $\mathbb{H}\times V_{\mathbb{H}}^R \ni (\mathrm{q}; \ ) \longmapsto \mathrm{q} \ \in V_{\mathbb{H}}^R$ given by

$$\mathrm{q} \ := \sum_{k\in N} '_k \mathrm{q}\langle '_k \mid \ \rangle; \tag{3.5}$$

for all $(\mathrm{q}; \ ) \in \mathbb{H}\times V_{\mathbb{H}}^R$.

Proposition 3.7 ([6]). *The left product defined in (3.5) satisfies the following properties. For every* $;\ \in V_{\mathbb{H}}^R$ *and* $\mathrm{p};\mathrm{q}\in\mathbb{H}$,

(a) $\mathrm{q}(\ + \ ) = \mathrm{q} \ + \mathrm{q}$ *and* $\mathrm{q}(\ \mathrm{p}) = (\mathrm{q} \ )\mathrm{p}$.
(b) $\|\mathrm{q} \ \| = |\mathrm{q}|\| \ \|$.
(c) $\mathrm{q}(\mathrm{p} \ ) = (\mathrm{qp}) \ $.
(d) $\langle \overline{\mathrm{q}} \ \mid \ \rangle = \langle \ \mid \mathrm{q} \ \rangle$.
(e) $r \ = \ r$, *for all* $r\in\mathbb{R}$.
(f) $\mathrm{q}'_k = '_k\mathrm{q}$, *for all* $k\in N$.

Remark 3.8. (1) In more rigorous terms, instead of p we will write $\mathrm{p}\cdot$, because the notation from (3.5) may be confusing, when $V_{\mathbb{H}}^R = \mathbb{H}$. However, regarding the field $\mathbb{H}$ itself as a right $\mathbb{H}$-Hilbert space, an orthonormal basis $\mathcal{O}$ should consist only of a singleton, say $\{'_0\}$, with $|'_0| = 1$, because we clearly have $\ = '_0\langle '_0 \mid \ \rangle$, for all $\ \in\mathbb{H}$. The equality from (f) of Proposition 3.7 can be written as $\mathrm{p}'_0 = '_0\mathrm{p}$, for all $\mathrm{p}\in\mathbb{H}$. In fact, the left hand may be confusing and it should be understood as $\mathrm{p}\cdot '_0$, because the true equality $\mathrm{p}'_0 = '_0\mathrm{p}$ would imply that $'_0 = \pm 1$. For simplicity, we are writing p instead of writing $\mathrm{p}\cdot$.
(2) Also one can trivially see that $(\mathrm{p}+\mathrm{q}) \ = \mathrm{p} \ + \mathrm{q}$, for all $\mathrm{p};\mathrm{q}\in\mathbb{H}$ and $\ \in V_{\mathbb{H}}^R$.

Furthermore, the quaternionic left scalar multiplication of linear operators is also defined in [6,9]. For any fixed $\mathrm{q}\in\mathbb{H}$ and a given right linear operator $A:\mathcal{D}(A)\longrightarrow V_{\mathbb{H}}^R$, the left scalar multiplication of A is defined as a map $\mathrm{q}A:\mathcal{D}(A)\longrightarrow V_{\mathbb{H}}^R$ by the setting

$$(\mathrm{q}A) \ := \mathrm{q}(A \ ) = \sum_{k\in N} '_k\mathrm{q}\langle '_k \mid A \ \rangle; \tag{3.6}$$

for all $\ \in D(A)$. It is straightforward that $\mathrm{q}A$ is a right linear operator. If $\mathrm{q} \ \in\mathcal{D}(A)$, for all $\ \in\mathcal{D}(A)$, one can define right scalar multiplication of the right linear operator $A:\mathcal{D}(A)\longrightarrow V_{\mathbb{H}}^R$ as a map $A\mathrm{q}:\mathcal{D}(A)\longrightarrow V_{\mathbb{H}}^R$ by the setting

$$(A\mathrm{q}) \ := A(\mathrm{q} \ ); \tag{3.7}$$

for all $\ \in D(A)$. It is also right linear operator. One can easily obtain that, if $\mathrm{q} \ \in\mathcal{D}(A)$, for all $\ \in\mathcal{D}(A)$ and $\mathcal{D}(A)$ is dense in $V_{\mathbb{H}}^R$, then

$$(\mathrm{q}A)^\dagger = A^\dagger\overline{\mathrm{q}} \ \text{ and } \ (A\mathrm{q})^\dagger = \overline{\mathrm{q}}A^\dagger: \tag{3.8}$$

Proposition 3.9 ([10]). *Let* $A:\mathcal{D}(A)\subseteq V_{\mathbb{H}}^R\longrightarrow V_{\mathbb{H}}^R$ *be a densely defined right linear symmetric operator with the property that* $\mathbf{i} \ ; \mathbf{j} \ ; \mathbf{k} \ \in\mathcal{D}(A)$, *for all* $\ \in\mathcal{D}(A)$ *and* $\mathrm{q} = q_0+\mathbf{i}q_1+\mathbf{j}q_2+\mathbf{k}q_3\in\mathbb{H}$.

(a) *If* $\mathbf{i}A$, $\mathbf{j}A$ *and* $\mathbf{k}A$ *are anti-symmetric, then* $(\mathrm{q}A)^\dagger = \overline{\mathrm{q}}A$, $\mathrm{q}A = A\mathrm{q}$ *and*

$$\|(A-\mathrm{q}\mathbb{I}_{V_{\mathbb{H}}^R}) \ \|^2 = \|(A-q_0\mathbb{I}_{V_{\mathbb{H}}^R}) \ \|^2 + (q_1^2+q_2^2+q_3^2)\| \ \|^2;$$

for all $\ \in\mathcal{D}(A)$.

(b) *If* $\mathrm{q}A$ *is anti-symmetric, then*

$$\|(A-\overline{\mathrm{q}}\mathbb{I}_{V_{\mathbb{H}}^R}) \ \|^2 = \|A \ \|^2 + |\mathrm{q}|^2\| \ \|^2$$

for all $\ \in\mathcal{D}(A)$.

(c) *If* $\overline{\mathrm{q}}A$ *is anti-symmetric, then*

$$\|(A-\mathrm{q}\mathbb{I}_{V_{\mathbb{H}}^R}) \ \|^2 = \|A \ \|^2 + |\mathrm{q}|^2\| \ \|^2$$

for all $\ \in\mathcal{D}(A)$.

Proposition 3.10 ([10]). *Let* $A:\mathcal{D}(A)\subseteq V_{\mathbb{H}}^R\longrightarrow V_{\mathbb{H}}^R$ *be a densely defined symmetric right linear operator with the property that* $\mathbf{i} \ ; \mathbf{j} \ ; \mathbf{k} \ \in\mathcal{D}(A)$, *for all* $\ \in\mathcal{D}(A)$. *If the operators* $\mathbf{i}A$, $\mathbf{j}A$ *and* $\mathbf{k}A$ *are anti-symmetric, then for any* $\mathrm{q} = q_0+\mathbf{i}q_1+\mathbf{j}q_2+\mathbf{k}q_3\in\mathbb{H}$ *with* $q_1^2+q_2^2+q_3^2\neq 0$, *the following statements are equivalent:*

(a) *A is self-adjoint.*
(b) *A is closed and* $\ker(A^\dagger-\mathrm{q}\mathbb{I}_{V_{\mathbb{H}}^R}) = \{0\}$ *and* $\ker(A^\dagger-\overline{\mathrm{q}}\mathbb{I}_{V_{\mathbb{H}}^R}) = \{0\}$.
(c) $ran(A-\mathrm{q}\mathbb{I}_{V_{\mathbb{H}}^R}) = ran(A-\overline{\mathrm{q}}\mathbb{I}_{V_{\mathbb{H}}^R}) = V_{\mathbb{H}}^R$.

3.3. Regular points and defect numbers of quaternionic linear operators

Definition 3.11. Let $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ be a right linear operator. A quaternion $\mathfrak{q}$ is called a *regular point* for A if there exists a number $c_{\mathfrak{q}} > 0$ such that

$$\|(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R}) \quad \| \geq c_{\mathfrak{q}} \| \quad \|; \tag{3.9}$$

for all $\in \mathcal{D}(A)$. The set of regular points of A is the *regular domain* of A and it is denoted by (A).

Definition 3.12. Let $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ be a right linear operator. For $\mathfrak{q} \in \quad (A)$, we call the linear subspace $\operatorname{ran}(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})^{\perp}$ of $V_{\mathbb{H}}^R$ the *deficiency subspace* of A at $\mathfrak{q}$ and its dimension $d_{\mathfrak{q}}(A) = \dim \operatorname{ran}(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})^{\perp}$ is the *defect number* of A at $\mathfrak{q}$.

The following proposition discusses some basic properties of the above definitions:

Proposition 3.13. *Let $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ be a densely defined right linear operator and $\mathfrak{q} \in \mathbb{H}$.*

(a) *If $\mathfrak{q}_0 \in \quad (A)$ such that $|\mathfrak{q} - \mathfrak{q}_0| < c_{\mathfrak{q}_0}$, where $c_{\mathfrak{q}_0}$ is a constant satisfying (3.9), then $\mathfrak{q} \in \quad (A)$. That is, (A) is an open subset of $\mathbb{H}$.*

(b) *If A is closable, then $(\overline{A}) = \quad (A)$, $d_{\mathfrak{q}}(\overline{A}) = d_{\mathfrak{q}}(A)$ and*

$$ran(\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R}) = \overline{ran(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})};$$

for all $\mathfrak{q} \in \quad (A)$.

(c) *If A is closed and $\mathfrak{q} \in \quad (A)$, then $ran(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})$ is a closed linear subspace of $V_{\mathbb{H}}^R$.*

Proof. (a) Suppose that $\mathfrak{q}_0 \in \quad (A)$ such that $|\mathfrak{q} - \mathfrak{q}_0| < c_{\mathfrak{q}_0}$ where $c_{\mathfrak{q}_0}$ is a constant satisfying (3.9). Let $\in \mathcal{D}(A)$, then we have

$$\begin{aligned}\|(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R}) \quad \| = \|(A - \mathfrak{q}_0\mathbb{I}_{V_{\mathbb{H}}^R}) \quad - (\mathfrak{q} - \mathfrak{q}_0) \quad \| &\geq \|(A - \mathfrak{q}_0\mathbb{I}_{V_{\mathbb{H}}^R}) \quad \| - |\mathfrak{q} - \mathfrak{q}_0| \| \quad \| \\ &\geq (c_{\mathfrak{q}_0} - |\mathfrak{q} - \mathfrak{q}_0|) \| \quad \|:\end{aligned}$$

Thus $\mathfrak{q} \in \quad (A)$ as $|\mathfrak{q} - \mathfrak{q}_0| < c_{\mathfrak{q}_0}$. From this, we can say that $(A) \subseteq \quad (A)^{\circ}$-interior set of (A). Therefore (a) follows.

(b) Let $\in \overline{\operatorname{ran}(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})}$. Then there exists a sequence $\{ \quad _n\} \subset \operatorname{ran}(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})$ such that $_n \longrightarrow$ as $n \longrightarrow \infty$. Now for each $n \in \mathbb{N}$, there exists $_n \in \mathcal{D}(A)$ such that $_n = (A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R}) \quad _n$. By (3.9), we have, for any $m; n \in \mathbb{N}$

$$\| \quad _m - \quad _n\| \leq c_{\mathfrak{q}}^{-1}\|(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})(\quad _m - \quad _n)\| = c_{\mathfrak{q}}^{-1}\| \quad _m - \quad _n\|:$$

From this, $\{ \quad _n\}$ is a Cauchy sequence, because $\{ \quad _n\}$ is a Cauchy sequence. Then $\{ \quad _n\}$ is convergent and take $= \lim_n \quad _n$. Since $\lim_n A \quad _n = \lim_n(\quad _n + \mathfrak{q} \quad _n) = \quad + \mathfrak{q}$ and A is closable, we have $\in \mathcal{D}(\overline{A})$ and $\overline{A} \quad = \quad + \mathfrak{q}$. Thus $= (\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R}) \quad \in \operatorname{ran}(\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})$. This proves that $\overline{\operatorname{ran}(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})} \subseteq \operatorname{ran}(\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})$. On the other hand, take $\in \operatorname{ran}(\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})$, then there exists $\in \mathcal{D}(\overline{A})$ such that $= (\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})$. By the definition of $\overline{A}$, we have for all $\{ \quad _n\} \subset \mathcal{D}(A)$ with $_n \longrightarrow$, $A \quad _n \longrightarrow \overline{A}$. From this, the converse inclusion follows immediately. Therefore,

$$\operatorname{ran}(\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R}) = \overline{\operatorname{ran}(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})}:$$

Let $\mathfrak{q} \in \quad (A)$, then there exists a number $c_{\mathfrak{q}} > 0$ such that

$$\|(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R}) \quad \| \geq c_{\mathfrak{q}} \| \quad \|;$$

for all $\in \mathcal{D}(A)$. Now take $\in \mathcal{D}(\overline{A})$, then $A \quad _n \longrightarrow \overline{A}$, for all $\{ \quad _n\} \subset \mathcal{D}(A)$ with $_n \longrightarrow$, and for each $n \in \mathbb{N}$,

$$\|(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R}) \quad _n\| \geq c_{\mathfrak{q}} \| \quad _n\|:$$

Taking limit $n \longrightarrow \infty$ on both sides, we get

$$\|(\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R}) \quad \| \geq c_{\mathfrak{q}} \| \quad \|;$$

because $\mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R} \quad _n = \sum_{k \in N} \varphi_k \mathfrak{q} \langle \varphi_k \mid \quad _n\rangle \longrightarrow \sum_{k \in N} \varphi_k \mathfrak{q} \langle \varphi_k \mid \quad \rangle = \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R}$. This proves that $(A) \subseteq \quad (\overline{A})$ and the converse inclusion immediately follows from the fact that $\mathcal{D}(A) \subseteq \mathcal{D}(\overline{A})$. Thus $(A) = \quad (\overline{A})$.

Since $\operatorname{ran}(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R}) \subseteq \operatorname{ran}(\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})$, it follows that $\operatorname{ran}(\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})^{\perp} \subseteq \operatorname{ran}(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})^{\perp}$: Let us now take $\in \operatorname{ran}(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})^{\perp}$. Let $\in \operatorname{ran}(\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})$, then there exists $\in \mathcal{D}(\overline{A})$ such that $= (\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})$ and $A \quad _n \longrightarrow \overline{A}$, for all $\{ \quad _n\} \subset \mathcal{D}(A)$ with $_n \longrightarrow$. If we take $_n = (A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R}) \quad _n$, for all $n \in \mathbb{N}$, then $\langle \quad \mid \quad _n\rangle = 0$, for all $n \in \mathbb{N}$. But $\langle \quad \mid \quad _n\rangle \longrightarrow \langle \quad \mid \quad \rangle$. Thus $\langle \quad \mid \quad \rangle = 0$. That is, $\in \operatorname{ran}(\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})^{\perp}$. Therefore $\operatorname{ran}(\overline{A} - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})^{\perp} = \operatorname{ran}(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})^{\perp}$. Hence $d_{\mathfrak{q}}(\overline{A}) = d_{\mathfrak{q}}(A)$ and point (b) follows.

(c) Since A is closed, we have $A = \overline{A}$. Thus by the result (b), it immediately follows that $\operatorname{ran}(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^R})$ is a closed subspace. Hence the result (c) follows.

3.4. S-spectrum of unbounded quaternionic linear Operators

For a given right linear operator $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^{R} \longrightarrow V_{\mathbb{H}}^{R}$ and $\mathfrak{q} \in \mathbb{H}$, we define the operator $Q_{\mathfrak{q}}(A) : \mathcal{D}(A^2) \longrightarrow \mathbb{H}$ by

$$Q_{\mathfrak{q}}(A) = A^2 - 2\mathrm{Re}(\mathfrak{q})A + |\mathfrak{q}|^2 \mathbb{I}_{V_{\mathbb{H}}^{R}};$$

where $\mathfrak{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ is a quaternion, $\mathrm{Re}(\mathfrak{q}) = q_0$ and $|\mathfrak{q}|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$: In the literature, the operator $Q_{\mathfrak{q}}(A)$ is sometimes also denoted by $R_{\mathfrak{q}}(A)$ and it is called pseudo-resolvent since it is not the resolvent operator of A but it is the one related to the notion of spectrum as we shall see in the next definition. The notion of S-spectrum has been introduced by one of the authors and her collaborators. For more information, the reader may consult e.g. [9,11,12,6].

Definition 3.14. Let $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^{R} \longrightarrow V_{\mathbb{H}}^{R}$ be a right linear operator. The *S-resolvent set* (also called *spherical resolvent* set) of A is the set $\;_S(A)(\subset \mathbb{H})$ such that the three following conditions hold true:

(a) $\ker(Q_{\mathfrak{q}}(A)) = \{0\}$.
(b) $\mathrm{ran}(Q_{\mathfrak{q}}(A))$ is dense in $V_{\mathbb{H}}^{R}$.
(c) $Q_{\mathfrak{q}}(A)^{-1} : \mathrm{ran}(Q_{\mathfrak{q}}(A)) \longrightarrow \mathcal{D}(A^2)$ is bounded.

The *S-spectrum* (also called *spherical spectrum*) $\;_S(A)$ of A is defined by setting $\;_S(A) := \mathbb{H} \setminus \;_S(A)$. It decomposes into three disjoint subsets as follows:

(i) the *spherical point spectrum* of A:

$$_{pS}(A) := \{\mathfrak{q} \in \mathbb{H} \mid \ker(Q_{\mathfrak{q}}(A)) \neq \{0\}\}:$$

(ii) the *spherical residual spectrum* of A:

$$_{rS}(A) := \{\mathfrak{q} \in \mathbb{H} \mid \ker(Q_{\mathfrak{q}}(A)) = \{0\}; \overline{\mathrm{ran}(Q_{\mathfrak{q}}(A))} \neq V_{\mathbb{H}}^{R}\}:$$

(iii) the *spherical continuous spectrum* of A:

$$_{cS}(A) := \{\mathfrak{q} \in \mathbb{H} \mid \ker(Q_{\mathfrak{q}}(A)) = \{0\}; \overline{\mathrm{ran}(Q_{\mathfrak{q}}(A))} = V_{\mathbb{H}}^{R}; Q_{\mathfrak{q}}(A)^{-1} \notin \mathcal{B}(V_{\mathbb{H}}^{R})\}:$$

If $A \quad = \quad \mathfrak{q}$ for some $\mathfrak{q} \in \mathbb{H}$ and $\quad \in V_{\mathbb{H}}^{R} \setminus \{0\}$, then is called an *eigenvector of A with right eigenvalue* $\mathfrak{q}$. The set of right eigenvalues coincides with the point S-spectrum, see [6], Proposition 4.5.

Proposition 3.15 ([6,8]). *Let $A \in \mathcal{L}(V_H^R)$ and A be self-adjoint, then $\;_S(A) \subset \mathbb{R}$.*

Proposition 3.16. *Let $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^{R} \longrightarrow V_{\mathbb{H}}^{R}$ be right linear operator with the property that $\mathbf{i}\;;\;\mathbf{j}\;;\;\mathbf{k}\quad \in \mathcal{D}(A)$, for all $\quad \in \mathcal{D}(A)$, and $\mathfrak{q} \in \mathbb{H}$. Then the pseudo-resolvent operator $Q_{\mathfrak{q}}(A)$ of A can be written as follows:*

$$Q_{\mathfrak{q}}(A) = \frac{1}{2}\left[(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^{R}})(A - \overline{\mathfrak{q}}\mathbb{I}_{V_{\mathbb{H}}^{R}}) + (A - \overline{\mathfrak{q}}\mathbb{I}_{V_{\mathbb{H}}^{R}})(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^{R}})\right]: \tag{3.10}$$

Furthermore, if A is densely defined, closed and symmetric with $\overline{\mathcal{D}(A^2)} = V_{\mathbb{H}}^{R}$, then $\;(A) \subseteq \;_S(A)$.

Proof. Formula (3.10) has been proved in [13], Proposition 5.9. To prove the second part of the statement, suppose that A is a densely defined closed symmetric operator then, using Cauchy–Schwarz inequality, we have for any $\quad \in \mathcal{D}(A^2)$,

$$\begin{aligned}\|Q_{\mathfrak{q}}(A)\quad\|\|\quad\| &\geq \langle Q_{\mathfrak{q}}(A)\quad \mid \quad\rangle \\ &= \frac{1}{2}\langle (A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^{R}})(A - \overline{\mathfrak{q}}\mathbb{I}_{V_{\mathbb{H}}^{R}})\quad \mid \quad\rangle + \frac{1}{2}\langle (A - \overline{\mathfrak{q}}\mathbb{I}_{V_{\mathbb{H}}^{R}})(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^{R}})\quad \mid \quad\rangle \\ &= \frac{1}{2}\|(A - \overline{\mathfrak{q}}\mathbb{I}_{V_{\mathbb{H}}^{R}})\quad\|^2 + \frac{1}{2}\|(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^{R}})\quad\|^2:\end{aligned}$$

That is, for any $\mathfrak{q} \in \mathbb{H}$ and $\quad \in \mathcal{D}(A^2)$,

$$\|Q_{\mathfrak{q}}(A)\quad\|\|\quad\| \geq \frac{1}{2}\|(A - \overline{\mathfrak{q}}\mathbb{I}_{V_{\mathbb{H}}^{R}})\quad\|^2 + \frac{1}{2}\|(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^{R}})\quad\|^2: \tag{3.11}$$

Thus, if $\mathfrak{q} \in \;(A)$, then there exists $c_{\mathfrak{q}} > 0$ such that (3.9) holds. Let $\quad \in \mathcal{D}(A^2)$, then

$$\|Q_{\mathfrak{q}}(A)\quad\|\|\quad\| \geq \frac{1}{2}\|(A - \mathfrak{q}\mathbb{I}_{V_{\mathbb{H}}^{R}})\quad\|^2 \geq \frac{c_{\mathfrak{q}}^2}{2}\|\quad\|^2:$$

This is equivalent to say that for every $\quad \in \mathcal{D}(A^2)$,

$$\|Q_{\mathfrak{q}}(A)\quad\| \geq \frac{c_{\mathfrak{q}}^2}{2}\|\quad\|: \tag{3.12}$$