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Berberian Extension and its S-spectra in a Quaternionic Hilbert Space

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Abstract. For a bounded right linear operators A, in a right quaternionic Hilbert space $V_{\mathbb{H}}^R$, following the complex formalism, we study the Berberian extension A°, which is an extension of A in a right quaternionic Hilbert space obtained from $V_{\mathbb{H}}^R$. In the complex setting, the important feature of the Berberian extension is that it converts approximate point spectrum of A into point spectrum of A°. We show that the same is true for the quaternionic S-spectrum. As in the complex case, we use the Berberian extension to study some properties of the commutator of two quaternionic bounded right linear operators.

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1. Introduction

In 1962 Berberian extended a bounded linear operator A on a complex Hilbert space X to an operator A° on a complex Hilbert space obtained from X. An important feature of this extension is that it converts approximate point spectrum of A into point spectrum of A° [3]. This extension is also a useful tool in studying the spectrum of commutator of two bounded linear operators [11].

In the complex theory this extension goes as follows. Let X be a complex Hilbert space. Let $l^{\infty}(X)$ denotes the space of all bounded sequence of elements of X, and let $c_0(X)$ denote the space of all null sequences in X. Endowed with the canonical norm, the space $\mathfrak{X} = l^{\infty}(X)/c_0(X)$ is a Hilbert space into which X can be isometrically embedded. Every operator

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$A \in B(X)$, the set of all bounded linear operators on X, defines by component wise action an operator on $l^{\infty}(X)$ which leaves $c_0(X)$ invariant, and hence induces an operator $A^{\circ} \in B(\mathfrak{X})$. It is immediate that A° is an extension of A when X is regarded as a subspace of $\mathfrak{X}$, and that the mapping that assigns to each $A \in B(X)$ its Berberian extension $A^{\circ} \in B(\mathfrak{X})$ is an isometric algebra homomorphism.

In this note we shall study the Berberian extension of a quaternionic right linear operator A on a right quaternionic Hilbert space and show that the approximate point S-spectrum of A coincides with the point S-spectrum of the Berberian extension A°. Following the complex formalism given in [11], we shall also study certain S-spectral properties of the commutator of two quaternionic bounded right linear operators.

In the complex setting, in a complex Hilbert space or Banach space $\mathfrak{H}$, for a bounded linear operator, A, the spectrum is defined as the set of complex numbers λ for which the operator $Q_{\lambda}(A) = A - \lambda\mathbb{I}_{\mathfrak{H}}$, where $\mathbb{I}_{\mathfrak{H}}$ is the identity operator on $\mathfrak{H}$, is not invertible. In the quaternionic setting, let $V_{\mathbb{H}}^{R}$ be a separable right quaternionic Hilbert space or Banach space, A be a bounded right linear operator, and $R_{\mathfrak{q}}(A) = A^2 - 2\text{Re}(\mathfrak{q})A + |\mathfrak{q}|^2\mathbb{I}_{V_{\mathbb{H}}^{R}}$, with $\mathfrak{q} \in \mathbb{H}$, the set of all quaternions, be the pseudo-resolvent operator. The S-spectrum is defined as the set of quaternions $\mathfrak{q}$ for which $R_{\mathfrak{q}}(A)$ is not invertible. The notion of S-spectrum was introduced in 2006 by Colombo and Sabadini. The discovery and the importance of this spectrum is well explained in [6]. Further developments on the theory of S-spectrum can be found in the book [7]. In the complex case various classes of spectra, such as approximate point spectrum, surjectivity spectrum etc. are defined by placing restrictions on the operator $Q_{\lambda}(A)$. In this regard, in the quaternionic setting, these spectra are also defined by placing the same restrictions to the operator $R_{\mathfrak{q}}(A)$ [12,14].

Due to the non-commutativity, in the quaternionic case there are three types of Hilbert spaces: left, right, and two-sided, depending on how vectors are multiplied by scalars. This fact can entail several problems. For example, when a Hilbert space $\mathcal{H}$ is one-sided (either left or right) the set of linear operators acting on it does not have a linear structure. Moreover, in a one sided quaternionic Hilbert space, given a linear operator A and a quaternion $\mathfrak{q} \in \mathbb{H}$, in general we have that $(\mathfrak{q}A)^{\dagger} \neq \overline{\mathfrak{q}}A^{\dagger}$ (see [13] for details). These restrictions can severely prevent the generalization to the quaternionic case of results valid in the complex setting. Even though most of the linear spaces are one-sided, it is possible to introduce a notion of multiplication on both sides by fixing an arbitrary Hilbert basis of $\mathcal{H}$. This fact allows to have a linear structure on the set of linear operators, which is a minimal requirement to develop a full theory.

2. Mathematical Preliminaries

In order to make the paper self-contained, we recall some facts about quaternions which may not be well-known. For details we refer the reader to [1,10, 15].

2.1. Quaternions

Let $\mathbb{H}$ denote the field of all quaternions and $\mathbb{H}^*$ the group (under quaternionic multiplication) of all invertible quaternions. A general quaternion can be written as

$$\mathfrak{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \quad q_0, q_1, q_2, q_3 \in \mathbb{R},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the three quaternionic imaginary units, satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$. The quaternionic conjugate of $\mathfrak{q}$ is

$$\overline{\mathfrak{q}} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3,$$

while $|\mathfrak{q}| = (\mathfrak{q}\overline{\mathfrak{q}})^{1/2}$ denotes the usual norm of the quaternion $\mathfrak{q}$. If $\mathfrak{q}$ is non-zero element, it has inverse $\mathfrak{q}^{-1} = \dfrac{\overline{\mathfrak{q}}}{|\mathfrak{q}|^2}$.

2.2. Quaternionic Hilbert Spaces

In this subsection we discuss right quaternionic Hilbert spaces. For more details we refer the reader to [1,10,15].

2.2.1. Right Quaternionic Hilbert Space. Let $V_{\mathbb{H}}^R$ be a vector space under right multiplication by quaternions. For $\phi, \psi, \omega \in V_{\mathbb{H}}^R$ and $\mathfrak{q} \in \mathbb{H}$, the inner product

$$\langle \cdot \mid \cdot \rangle_{V_{\mathbb{H}}^R} : V_{\mathbb{H}}^R \times V_{\mathbb{H}}^R \longrightarrow \mathbb{H}$$

satisfies the following properties

(i) $\overline{\langle \phi \mid \psi \rangle_{V_{\mathbb{H}}^R}} = \langle \psi \mid \phi \rangle_{V_{\mathbb{H}}^R}$
(ii) $\|\phi\|^2_{V_{\mathbb{H}}^R} = \langle \phi \mid \phi \rangle_{V_{\mathbb{H}}^R} > 0$ unless $\phi = 0$, a real norm
(iii) $\langle \phi \mid \psi + \omega \rangle_{V_{\mathbb{H}}^R} = \langle \phi \mid \psi \rangle_{V_{\mathbb{H}}^R} + \langle \phi \mid \omega \rangle_{V_{\mathbb{H}}^R}$
(iv) $\langle \phi \mid \psi\mathfrak{q} \rangle_{V_{\mathbb{H}}^R} = \langle \phi \mid \psi \rangle_{V_{\mathbb{H}}^R}\mathfrak{q}$
(v) $\langle \phi\mathfrak{q} \mid \psi \rangle_{V_{\mathbb{H}}^R} = \overline{\mathfrak{q}}\langle \phi \mid \psi \rangle_{V_{\mathbb{H}}^R}$

where $\overline{\mathfrak{q}}$ stands for the quaternionic conjugate. It is always assumed that the space $V_{\mathbb{H}}^R$ is complete under the norm given above and separable. Then, together with $\langle \cdot \mid \cdot \rangle_{V_{\mathbb{H}}^R}$ this defines a right quaternionic Hilbert space. Quaternionic Hilbert spaces share many of the standard properties of complex Hilbert spaces. Every separable quaternionic Hilbert space posses a basis. It should be noted that once a Hilbert basis is fixed, every left (resp. right) quaternionic Hilbert space also becomes a right (resp. left) quaternionic Hilbert space [10,15].

The field of quaternions $\mathbb{H}$ itself can be turned into a left quaternionic Hilbert space by defining the inner product $\langle \mathfrak{q} \mid \mathfrak{q}' \rangle = \mathfrak{q}\overline{\mathfrak{q}'}$ or into a right quaternionic Hilbert space with $\langle \mathfrak{q} \mid \mathfrak{q}' \rangle = \overline{\mathfrak{q}}\mathfrak{q}'$.

3. Right Quaternionic Linear Operators and Some Basic Properties

In this section we shall define right $\mathbb{H}$-linear operators and recall some basis properties. Most of them are very well known. In this manuscript, we follow the notations in [2,10]. We shall also recall some results pertinent to the development of the paper.

Definition 3.1. A mapping $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$, where $\mathcal{D}(A)$ stands for the domain of A, is said to be right $\mathbb{H}$-linear operator or, for simplicity, right linear operator, if

$$A(\phi\mathfrak{a} + \psi\mathfrak{b}) = (A\phi)\mathfrak{a} + (A\psi)\mathfrak{b}, \text{ if } \phi, \psi \in \mathcal{D}(A) \text{ and } \mathfrak{a}, \mathfrak{b} \in \mathbb{H}.$$

The set of all right linear operators from $V_{\mathbb{H}}^R$ to $V_{\mathbb{H}}^R$ will be denoted by $\mathcal{L}(V_{\mathbb{H}}^R)$ and the identity linear operator on $V_{\mathbb{H}}^R$ will be denoted by $\mathbb{I}_{V_{\mathbb{H}}^R}$. For a given $A \in \mathcal{L}(V_{\mathbb{H}}^R)$, the range and the kernel will be

$$\begin{aligned} \operatorname{ran}(A) &= \{\psi \in V_{\mathbb{H}}^R \mid A\phi = \psi \quad \text{for} \ \ \phi \in \mathcal{D}(A)\} \\ \ker(A) &= \{\phi \in \mathcal{D}(A) \mid A\phi = 0\}. \end{aligned}$$

We call an operator $A \in \mathcal{L}(V_{\mathbb{H}}^R)$ bounded if

$$\|A\| = \sup_{\|\phi\|_{V_{\mathbb{H}}^R} = 1} \|A\phi\|_{V_{\mathbb{H}}^R} < \infty, \tag{3.1}$$

or equivalently, there exist $K \geq 0$ such that $\|A\phi\|_{V_{\mathbb{H}}^R} \leq K\|\phi\|_{V_{\mathbb{H}}^R}$ for all $\phi \in \mathcal{D}(A)$. The set of all bounded right linear operators from $V_{\mathbb{H}}^R$ to $V_{\mathbb{H}}^R$ will be denoted by $B(V_{\mathbb{H}}^R)$.

Assume that $V_{\mathbb{H}}^R$ is a right quaternionic Hilbert space, A is a right linear operator acting on it. Then, there exists a unique linear operator $A^\dagger$ such that

$$\langle \psi \mid A\phi \rangle_{V_{\mathbb{H}}^R} = \langle A^\dagger\psi \mid \phi \rangle_{V_{\mathbb{H}}^R}; \quad \text{for all } \phi \in \mathcal{D}(A), \psi \in \mathcal{D}(A^\dagger), \tag{3.2}$$

where the domain $\mathcal{D}(A^\dagger)$ of $A^\dagger$ is defined by

$$\mathcal{D}(A^\dagger) = \{\psi \in V_{\mathbb{H}}^R \mid \exists\varphi \text{ such that } \langle \psi \mid A\phi \rangle_{V_{\mathbb{H}}^R} = \langle \varphi \mid \phi \rangle_{V_{\mathbb{H}}^R}\}.$$

Proposition 3.2. [10] *Let* $A, B \in B(V_{\mathbb{H}}^R)$ *then*

(a) $(A + B)^\dagger = A^\dagger + B^\dagger$.
(b) $(AB)^\dagger = B^\dagger A^\dagger$.

We shall need the following results which are already appeared in [10, 12].

Proposition 3.3. *Let* $A \in B(V_{\mathbb{H}}^R)$. *Then*

(a) $ran(A)^\perp = ker(A^\dagger)$.
(b) $ker(A) = ran(A^\dagger)^\perp$.
(c) $ker(A)$ *is closed subspace of* $V_{\mathbb{H}}^R$.

Theorem 3.4. [12] (Bounded inverse theorem) *Let* $A \in B(V_{\mathbb{H}}^R)$, *then the following results are equivalent.*

(a) A has a bounded inverse on its range.
(b) A is bounded below.
(c) A is injective and has a closed range.

Proposition 3.5. [12] *Let* $A \in \mathcal{B}(V_{\mathbb{H}}^R)$. *Then,*

(a) A is invertible if and only if it is injective with a closed range (i.e., $ker(A) = \{0\}$ and $\overline{ran(A)} = ran(A)$).
(b) A is left (right) invertible if and only if $A^{\dagger}$ is right (left) invertible.

Proposition 3.6. [12] *$A \in B(V_{\mathbb{H}}^{R})$ is surjective if and only if A is right invertible.*

Proposition 3.7. *$A \in B(V_{\mathbb{H}}^{R})$ is injective if and only if A is left invertible.*

Proof. From point (b) of Proposition 3.5, point (b) of Proposition 3.3, and Proposition 3.6, we have, A is left invertible $\Leftrightarrow$ $A^{\dagger}$ is right invertible $\Leftrightarrow$ $\text{ran}(A^{\dagger}) = V_{\mathbb{H}}^{R} \Leftrightarrow \ker(A) = \{0\}$. This completes the proof. $\square$

3.1. Left Scalar Multiplications on $V_{\mathbb{H}}^{R}$

We shall extract the definition and some properties of left scalar multiples of vectors on $V_{\mathbb{H}}^{R}$ from [10] as needed for the development of the manuscript. The left scalar multiple of vectors on a right quaternionic Hilbert space is an extremely non-canonical operation associated with a choice of preferred Hilbert basis. Since $V_{\mathbb{H}}^{R}$ is a separable Hilbert space, $V_{\mathbb{H}}^{R}$ has a Hilbert basis

$$\mathcal{O} = \{\varphi_k \mid k \in N\}, \tag{3.3}$$

where N is a countable index set. The left scalar multiplication on $V_{\mathbb{H}}^{R}$ induced by $\mathcal{O}$ is defined as the map $\mathbb{H} \times V_{\mathbb{H}}^{R} \ni (\mathfrak{q}, \phi) \longmapsto \mathfrak{q}\phi \in V_{\mathbb{H}}^{R}$ given by

$$\mathfrak{q}\phi := \sum_{k \in N} \varphi_k \mathfrak{q} \langle \varphi_k \mid \phi \rangle_{V_{\mathbb{H}}^{R}}, \tag{3.4}$$

for all $(\mathfrak{q}, \phi) \in \mathbb{H} \times V_{\mathbb{H}}^{R}$.

Proposition 3.8. [10] *The left product defined in the Eq. 3.4 satisfies the following properties. For every $\phi, \psi \in V_{\mathbb{H}}^{R}$ and $\mathfrak{p}, \mathfrak{q} \in \mathbb{H}$,*

(a) $\mathfrak{q}(\phi + \psi) = \mathfrak{q}\phi + \mathfrak{q}\psi$ and $\mathfrak{q}(\phi\mathfrak{p}) = (\mathfrak{q}\phi)\mathfrak{p}$.
(b) $\|\mathfrak{q}\phi\|_{V_{\mathbb{H}}^{R}} = |\mathfrak{q}|\|\phi\|_{V_{\mathbb{H}}^{R}}$.
(c) $\mathfrak{q}(\mathfrak{p}\phi) = (\mathfrak{q}\mathfrak{p})\phi$.
(d) $\langle \overline{\mathfrak{q}}\phi \mid \psi \rangle_{V_{\mathbb{H}}^{R}} = \langle \phi \mid \mathfrak{q}\psi \rangle_{V_{\mathbb{H}}^{R}}$.
(e) $r\phi = \phi r$, for all $r \in \mathbb{R}$.
(f) $\mathfrak{q}\varphi_k = \varphi_k \mathfrak{q}$, for all $k \in N$.

Remark 3.9. (1) The meaning of writing $\mathfrak{p}\phi$ is $\mathfrak{p} \cdot \phi$, because the notation from the Eq. 3.4 may be confusing, when $V_{\mathbb{H}}^{R} = \mathbb{H}$. However, regarding the field $\mathbb{H}$ itself as a right $\mathbb{H}$-Hilbert space, an orthonormal basis $\mathcal{O}$ should consist only of a singleton, say $\{\varphi_0\}$, with $\mid \varphi_0 \mid = 1$, because we clearly have $\theta = \varphi_0 \langle \varphi_0 \mid \theta \rangle$, for all $\theta \in \mathbb{H}$. The equality from (f) of Proposition 3.8 can be written as $\mathfrak{p}\varphi_0 = \varphi_0 \mathfrak{p}$, for all $\mathfrak{p} \in \mathbb{H}$. In fact, the left hand may be confusing and it should be understood as $\mathfrak{p} \cdot \varphi_0$, because the true equality $\mathfrak{p}\varphi_0 = \varphi_0 \mathfrak{p}$ would imply that $\varphi_0 = \pm 1$. For the simplicity, we are writing $\mathfrak{p}\phi$ instead of writing $\mathfrak{p} \cdot \phi$.

(2) Also one can trivially see that $(\mathfrak{p} + \mathfrak{q})\phi = \mathfrak{p}\phi + \mathfrak{q}\phi$, for all $\mathfrak{p}, \mathfrak{q} \in \mathbb{H}$ and $\phi \in V_{\mathbb{H}}^{R}$.

Furthermore, the quaternionic left scalar multiplication of linear operators is also defined in [5,10]. For any fixed $\mathfrak{q} \in \mathbb{H}$ and a given right linear operator $A : V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$, the left scalar multiplication of A is defined as a map $\mathfrak{q}A : V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ by the setting

$$(\mathfrak{q}A)\phi := \mathfrak{q}(A\phi) = \sum_{k\in N} \varphi_k \mathfrak{q} \langle \varphi_k \mid A\phi \rangle_{V_{\mathbb{H}}^R}, \tag{3.5}$$

for all $\phi \in V_{\mathbb{H}}^R$. It is straightforward that $\mathfrak{q}A$ is a right linear operator. We can define right scalar multiplication of the right linear operator $A : V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ as a map $A\mathfrak{q} : V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ by the setting

$$(A\mathfrak{q})\phi := A(\mathfrak{q}\phi), \tag{3.6}$$

for all $\phi \in V_{\mathbb{H}}^R$. It is also right linear operator. One can easily see that

$$(\mathfrak{q}A)^\dagger = A^\dagger \overline{\mathfrak{q}} \text{ and } (A\mathfrak{q})^\dagger = \overline{\mathfrak{q}} A^\dagger. \tag{3.7}$$

3.2. S-spectrum

For a given right linear operator $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ and $\mathfrak{q} \in \mathbb{H}$, we define the operator $R_{\mathfrak{q}}(A) : \mathcal{D}(A^2) \longrightarrow \mathbb{H}$ by

$$R_{\mathfrak{q}}(A) = A^2 - 2\text{Re}(\mathfrak{q})A + |\mathfrak{q}|^2 \mathbb{I}_{V_{\mathbb{H}}^R},$$

where $\mathfrak{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ is a quaternion, $\text{Re}(\mathfrak{q}) = q_0$ and $|\mathfrak{q}|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$.

In the literature, the operator is called pseudo-resolvent since it is not the resolvent operator of A but it is the one related to the notion of spectrum as we shall see in the next definition. For more information, on the notion of S-spectrum the reader may consult e.g. [4,5,9,10].

Definition 3.10. Let $A : \mathcal{D}(A) \subseteq V_{\mathbb{H}}^R \longrightarrow V_{\mathbb{H}}^R$ be a right linear operator. The *S-resolvent set* (also called *spherical resolvent* set) of A is the set $\rho_S(A)$ ($\subset \mathbb{H}$) such that the three following conditions hold true:

(a) $\ker(R_{\mathfrak{q}}(A)) = \{0\}$.
(b) $\text{ran}(R_{\mathfrak{q}}(A))$ is dense in $V_{\mathbb{H}}^R$.
(c) $R_{\mathfrak{q}}(A)^{-1} : \text{ran}(R_{\mathfrak{q}}(A)) \longrightarrow \mathcal{D}(A^2)$ is bounded.

The *S-spectrum* (also called *spherical spectrum*) $\sigma_S(A)$ of A is defined by setting $\sigma_S(A) := \mathbb{H} \smallsetminus \rho_S(A)$. For a bounded linear operator A we can write the resolvent set as

$$\begin{aligned}\rho_S(A) &= \{\mathfrak{q} \in \mathbb{H} \mid R_{\mathfrak{q}}(A) \text{ has an inverse in } B(V_{\mathbb{H}}^R)\} \\ &= \{\mathfrak{q} \in \mathbb{H} \mid \ker(R_{\mathfrak{q}}(A)) = \{0\} \quad \text{and} \quad \text{ran}(R_{\mathfrak{q}}(A)) = V_{\mathbb{H}}^R\}\end{aligned}$$

and the spectrum can be written as

$$\begin{aligned}\sigma_S(A) &= \mathbb{H} \backslash \rho_S(A) \\ &= \{\mathfrak{q} \in \mathbb{H} \mid R_{\mathfrak{q}}(A) \text{ has no inverse in } B(V_{\mathbb{H}}^R)\} \\ &= \{\mathfrak{q} \in \mathbb{H} \mid \ker(R_{\mathfrak{q}}(A)) \neq \{0\} \quad \text{or} \quad \text{ran}(R_{\mathfrak{q}}(A)) \neq V_{\mathbb{H}}^R\}.\end{aligned}$$

The spectrum $\sigma_S(A)$ decomposes into three major disjoint subsets as follows:

(i) the *spherical point spectrum* of A:

$$\sigma_{pS}(A) := \{\mathfrak{q} \in \mathbb{H} \mid \ker(R_{\mathfrak{q}}(A)) \neq \{0\}\}.$$

(ii) the *spherical residual spectrum* of A:

$$\sigma_{rS}(A) := \{\mathfrak{q} \in \mathbb{H} \mid \ker(R_{\mathfrak{q}}(A)) = \{0\}, \overline{\operatorname{ran}(R_{\mathfrak{q}}(A))} \neq V_{\mathbb{H}}^R \}.$$

(iii) the *spherical continuous spectrum* of A:

$$\sigma_{cS}(A) := \{\mathfrak{q} \in \mathbb{H} \mid \ker(R_{\mathfrak{q}}(A)) = \{0\}, \overline{\operatorname{ran}(R_{\mathfrak{q}}(A))} = V_{\mathbb{H}}^R, R_{\mathfrak{q}}(A)^{-1} \notin B(V_{\mathbb{H}}^R) \}.$$

If $A\phi = \phi\mathfrak{q}$ for some $\mathfrak{q} \in \mathbb{H}$ and $\phi \in V_{\mathbb{H}}^R \smallsetminus \{0\}$, then ϕ is called an *eigenvector of A with right eigenvalue* $\mathfrak{q}$. The set of right eigenvalues coincides with the point S-spectrum, see [10], Proposition 4.5.

Proposition 3.11. [8,10] *For $A \in B(V_{\mathbb{H}}^R)$, the resolvent set $\rho_S(A)$ is a non-empty open set and the spectrum $\sigma_S(A)$ is a non-empty compact set.*

Remark 3.12. For $A \in B(V_{\mathbb{H}}^R)$, since $\sigma_S(A)$ is a non-empty compact set so is its boundary. That is, $\partial\sigma_S(A) = \partial\rho_S(A) \neq \emptyset$.

Proposition 3.13. [6] *Let $A \in B(V_{\mathbb{H}}^R)$. Then ker$(R_{\mathfrak{q}}(A)) \neq \{0\}$ if and only if $\mathfrak{q}$ is a right eigenvalue of A. In particular every right eigenvalue belongs to $\sigma_S(A)$.*

Definition 3.14. [12] Let $A \in B(V_{\mathbb{H}}^R)$. The *approximate S-point spectrum* of A, denoted by $\sigma_{ap}^S(A)$, is defined as

$$\sigma_{ap}^S(A) = \{\mathfrak{q} \in \mathbb{H} \mid \text{there is a sequence } \{\phi_n\}_{n=1}^{\infty} \text{ such that } \|\phi_n\| = 1 \text{ and } \|R_{\mathfrak{q}}(A)\phi_n\| \longrightarrow 0\}.$$

Proposition 3.15. [12] *Let $A \in B(V_{\mathbb{H}}^R)$, then $\sigma_{pS}(A) \subseteq \sigma_{ap}^S(A)$.*

Definition 3.16. [12,14] The spherical compression spectrum of an operator $A \in B(V_{\mathbb{H}}^R)$, denoted by $\sigma_c^S(A)$, is defined as

$$\sigma_c^S(A) = \left\{\mathfrak{q} \in \mathbb{H} \mid \operatorname{ran}(R_{\mathfrak{q}}(A)) \text{ is not dense in } V_{\mathbb{H}}^R\right\}.$$

Definition 3.17. [14] Let $A \in B(V_{\mathbb{H}}^R)$. The surjectivity S-spectrum of A is defined as

$$\sigma_{su}^S(A) = \left\{\mathfrak{q} \in \mathbb{H} \mid \operatorname{ran}(R_{\mathfrak{q}}(A) \neq V_{\mathbb{H}}^R\right\}.$$

Clearly we have

$$\sigma_c^S(A) \subseteq \sigma_{su}^S(A) \quad \text{and} \quad \sigma_S(A) = \sigma_{pS}(A) \cup \sigma_{su}^S(A). \tag{3.8}$$

Proposition 3.18. [12] *Let $A \in B(V_{\mathbb{H}}^R)$. Then A has the following properties.*

(a) $\sigma_{pS}(A) \subseteq \sigma_c^S(A^\dagger)$ *and* $\sigma_c^S(A) = \sigma_{pS}(A^\dagger)$.
(b) $\sigma_{su}^S(A) = \sigma_{ap}^S(A^\dagger)$ *and* $\sigma_{ap}^S(A) = \sigma_{su}^S(A^\dagger)$.
(c) $\sigma_S(A) = \sigma_S(A^\dagger)$.

Proposition 3.19. [12] *If $A \in B(V_{\mathbb{H}}^R)$ and $\mathfrak{q} \in \mathbb{H}$, then the following statements are equivalent.*

(a) $\mathfrak{q} \notin \sigma_{ap}^S(A)$.
(b) $ker(R_{\mathfrak{q}}(A)) = \{0\}$ *and* $ran(R_{\mathfrak{q}}(A))$ *is closed.*
(c) There exists a constant $c \in \mathbb{R}$, $c > 0$ *such that* $\|R_{\mathfrak{q}}(A)\phi\| \geq c\|\phi\|$ *for all* $\phi \in \mathcal{D}(A^2)$.

Theorem 3.20. [10] *Let* $V_{\mathbb{H}}^R$ *be a right quaternionic Hilbert space equipped with a left scalar multiplication. Then the set* $B(V_{\mathbb{H}}^R)$ *equipped with the point-wise sum, with the left and right scalar multiplications defined in Eqs.* 3.5 *and* 3.6*, with the composition as product, with the adjunction* $A \longrightarrow A^\dagger$*, as in* 3.2*, as* $^*-$ *involution and with the norm defined in* 3.1*, is a quaternionic two-sided Banach* C^**-algebra with unity* $\mathbb{I}_{V_{\mathbb{H}}^R}$.

One can observe that in the above theorem, if the left scalar multiplication is left out on $V_{\mathbb{H}}^R$, then $B(V_{\mathbb{H}}^R)$ becomes a real Banach C^*-algebra with unity $\mathbb{I}_{V_{\mathbb{H}}^R}$.

4. Berberian Extension in the Quaternionic Setting

Following the definition given in [3] for complex bounded sequences, we denote by glim a *Banach generalized limit* defined for bounded sequences $\{\mathfrak{q}_n\} \subseteq \mathbb{H}$ with the following properties. For $\mathfrak{q} \in \mathbb{H}$ and $\{\mathfrak{q}_n\}, \{\mathfrak{p}_n\} \subseteq \mathbb{H}$,
(a) $\text{glim}(\mathfrak{q}_n + \mathfrak{p}_n) = \text{glim}(\mathfrak{q}_n) + \text{glim}(\mathfrak{p}_n)$;
(b) $\text{glim}(\mathfrak{q}_n \mathfrak{q}) = \text{glim}(\mathfrak{q}_n)\, \mathfrak{q}$;
(c) $\text{glim}(\mathfrak{q}\mathfrak{q}_n) = \mathfrak{q}\, \text{glim}(\mathfrak{q}_n)$;
(d) $\text{glim}(\mathfrak{q}_n) = \lim_{n\to\infty} \mathfrak{q}_n$ whenever $\{\mathfrak{q}_n\}$ is convergent;
(e) $\text{glim}(\mathfrak{q}_n) \geq 0$ when $\{\mathfrak{q}_n\} \subseteq \mathbb{R}$ and $\mathfrak{q}_n \geq 0$ for all n.

glim defines a positive linear form on the vector space $\mathfrak{M}$ of all quaternionic bounded sequences and c_0 denotes the set of quaternionic null sequences, that is, sequences that converge to zero, and has the value 1 for the constant sequence $\{1\}$. From properties (a) and (e) of glim, $\text{glim}(\mathfrak{q}_n)$ is real whenever $\mathfrak{q}_n$ is real for all n. Hence $\text{glim}(\overline{\mathfrak{q}}_n) = \overline{\text{glim}(\mathfrak{q}_n)}$ for any bounded sequence $\{\mathfrak{q}_n\} \subseteq \mathbb{H}$.

4.1. An extension of $V_{\mathbb{H}}^R$

Let

$$\mathcal{B} = \left\{ s = \{\phi_n\} \mid \{\phi_n\} \subseteq V_{\mathbb{H}}^R,\ \|\phi_n\|_{V_{\mathbb{H}}^R} < \infty\ \forall n,\ \text{that is, } \{\|\phi_n\|_{V_{\mathbb{H}}^R}\} \in \mathfrak{M} \right\}.$$

If $s = \{\phi_n\}$ and $t = \{\psi_n\}$ write $s = t$ whenever $\phi_n = \psi_n$ for all n. Also

$$s + t = \{\phi_n + \psi_n\} \quad \text{and} \quad s\mathfrak{q} = \{\phi_n \mathfrak{q}\},$$

with these operations $\mathcal{B}$ becomes a quaternionic right linear vector space. The left scalar multiplication, on $\mathcal{B}$, is defined as the map $\mathbb{H} \times \mathcal{B} \ni (\mathfrak{q}, s) \longmapsto \mathfrak{q}s \in \mathcal{B}$ given by

$$\mathfrak{q}s := \{\mathfrak{q}\phi_n\}, \tag{4.1}$$

for all $(\mathfrak{q}, s) = (\mathfrak{q}, \{\phi_n\}) \in \mathbb{H} \times \mathcal{B}$, where for each $n \in \mathbb{N}, \mathfrak{q}\phi_n$ is given by the Definition 3.4. Suppose that $s = \{\phi_n\}, t = \{\psi_n\} \in \mathcal{B}$. Since

$$|\langle \phi_n | \psi_n \rangle_{V_{\mathbb{H}}^R}| \leq \|\phi_n\|_{V_{\mathbb{H}}^R}\, \|\psi_n\|_{V_{\mathbb{H}}^R}, \quad \text{for all } n,$$

it is permissible to define

$$\Phi(s,t)=\mathrm{glim}(\langle\phi_n|\psi_n\rangle_{V_{\mathbb{H}}^R}).$$

We have the following properties for Φ.

(a) Since $\langle\phi_n|\psi_n\rangle_{V_{\mathbb{H}}^R}=\overline{\langle\psi_n|\phi_n\rangle_{V_{\mathbb{H}}^R}}$, we have $\Phi(s,t)=\overline{\Phi(t,s)}$. That is, Φ is symmetric.

(b) Since $\langle\phi_n|\phi_n\rangle_{V_{\mathbb{H}}^R}\geq 0$ for all n, $\Phi(s,s)\geq 0$ for all $s\in\mathcal{B}$. That is, Φ is positive.

(c) Φ is a bilinear functional, in the sense that Φ is left-antilinear with respect to the first variable,

$$\Phi(r\mathfrak{p}+s\mathfrak{q},t)=\overline{\mathfrak{p}}\Phi(r,t)+\overline{\mathfrak{q}}\Phi(s,t),\quad \text{for all } \mathfrak{p},\mathfrak{q}\in\mathbb{H} \text{ and } r,s,t\in\mathcal{B},$$

and Φ is right-linear with respect to the second variable,

$$\Phi(s,r\mathfrak{p}+t\mathfrak{q})=\Phi(s,r)\mathfrak{p}+\Phi(s,t)\mathfrak{q},\quad \text{for all } \mathfrak{p},\mathfrak{q}\in\mathbb{H} \text{ and } r,s,t\in\mathcal{B}.$$

From the Schwarz's inequality we have

$$|\Phi(s,t)|^2\leq\Phi(s,s)\Phi(t,t).$$

Let

$$\mathfrak{N}=\{s\in\mathcal{B}\mid\Phi(s,s)=0\}=\{s\in\mathcal{B}\mid\Phi(s,t)=0\ \forall\ t\in\mathcal{B}\}.$$

Clearly $\mathfrak{N}$ is a right linear subspace of $\mathcal{B}$. Write $[s]=s+\mathfrak{N}$ for a coset. The quotient right linear vector space $\mathfrak{P}=\mathcal{B}/\mathfrak{N}$ becomes an inner product space by defining

$$\langle[s]\mid[t]\rangle_{\mathfrak{P}}=\Phi(s,t).$$

If $u=\{[\phi_n]\}=\{\phi_n\}+\mathfrak{N}$ and $v=\{[\psi_n]\}=\{\psi_n\}+\mathfrak{N}$, then

$$\langle u\mid v\rangle_{\mathfrak{P}}=\langle[\phi_n]\mid[\psi_n]\rangle_{\mathfrak{P}}=\mathrm{glim}\langle\phi_n\mid\psi_n\rangle_{V_{\mathbb{H}}^R}. \tag{4.2}$$

Using the left scalar multiplication defined on $\mathcal{B}$, by the Eq. 4.1, we can define a left scalar multiplication on $\mathfrak{P}$ by the map $\mathbb{H}\times\mathfrak{P}\ni(\mathfrak{q},s)\longmapsto\mathfrak{q}[s]\in\mathfrak{P}$ given by

$$\mathfrak{q}[s]:=\mathfrak{q}s+\mathfrak{N}, \tag{4.3}$$

for all $(\mathfrak{q},[s])=(\mathfrak{q},s+\mathfrak{N})\in\mathbb{H}\times\mathfrak{P}$. Following proposition provides some properties of the above defined left scalar multiplication:

Proposition 4.1. *The left product defined in the Eq. 4.3 satisfies the following properties. For every* $[s],[t]\in\mathfrak{P}$ *and* $\mathfrak{p},\mathfrak{q}\in\mathbb{H}$,

(a) $\mathfrak{q}([s]+[t])=\mathfrak{q}[s]+\mathfrak{q}[t]$ *and* $\mathfrak{q}([s]\mathfrak{p})=(\mathfrak{q}[s])\mathfrak{p}$.
(b) $\|\,\mathfrak{q}[s]\,\|_{\mathfrak{P}}=|\mathfrak{q}|\|\,[s]\,\|_{\mathfrak{P}}$.
(c) $\mathfrak{q}(\mathfrak{p}[s])=(\mathfrak{q}\mathfrak{p})[s]$.
(d) $\langle\overline{\mathfrak{q}}[s]\mid[t]\rangle_{\mathfrak{P}}=\langle[s]\mid\mathfrak{q}[t]\rangle_{\mathfrak{P}}$.
(e) $r[s]=[s]r$, *for all* $r\in\mathbb{R}$.

Proof. The proof immediately follows from the Proposition 3.8 together with the Eqs. 4.1 and 4.3. □

Let $\phi \in V_{\mathbb{H}}^R$, we write $\{\phi\}$ for the sequence all of whose terms are ϕ and ϕ' for the coset $\{[\phi]\} = \{\phi\} + \mathfrak{N}$. Evidently

$$\langle [\phi] \mid [\psi] \rangle_{\mathfrak{P}} = \langle \phi | \psi \rangle_{V_{\mathbb{H}}^R},$$

and $\phi \mapsto [\phi]$ is an isometric right linear mapping of $V_{\mathbb{H}}^R$ onto a closed linear subspace $V_{\mathbb{H}}^{R'}$ of $\mathfrak{P}$. Regard $\mathfrak{P}$ as a linear subspace of its Hilbert space completion $\mathfrak{H}$. Then $V_{\mathbb{H}}^{R'}$ is a closed linear subspace of $\mathfrak{H}$ and $\mathfrak{P}$ is a dense linear subspace of $\mathfrak{H}$.

4.2. A Representation of $B(V_{\mathbb{H}}^R)$

Every operator A in $V_{\mathbb{H}}^R$ determines an operator A° in $\mathfrak{H}$ as follows.

If $s = \{\phi_n\} \in \mathcal{B}$ then the relation $\|A\phi_n\|_{V_{\mathbb{H}}^R} \leq \|A\| \; \|\phi_n\|_{V_{\mathbb{H}}^R}$ shows that $\{A\phi_n\} \in \mathcal{B}$. Define

$$A_0 : \mathcal{B} \longrightarrow \mathcal{B} \quad \text{by} \quad A_0 s = \{A\phi_n\},$$

then A_0 is a right linear mapping such that

$$\Phi(A_0 s, A_0 s) \leq \|A\| \Phi(s, s).$$

In particular, if $s \in \mathfrak{N}$, that is $\Phi(s, s) = 0$, then $A_0 s \in \mathfrak{N}$. it follows that

$$A^\circ : \mathfrak{P} \longrightarrow \mathfrak{P} \quad \text{by} \quad \{[\phi_n]\} \mapsto \{[A\phi_n]\} \tag{4.4}$$

is a well-defined right linear map. Thus

$$A^\circ s' = (A_0 s)'$$

and the inequality

$$\langle A^\circ u | A^\circ u \rangle_{\mathfrak{P}} \leq \|A\|^2 \langle u | u \rangle_{\mathfrak{P}}$$

is valid for all $u \in \mathfrak{P}$. That is, $\|A^\circ u\|_{\mathfrak{P}} \leq \|A\| \|u\|_{\mathfrak{P}}$, for all $u \in \mathfrak{P}$. Hence A° is bounded (continuous), and $\|A^\circ\|_\circ \leq \|A\|$, $\|\cdot\|_\circ$ is the norm on $B(\mathfrak{H})$. The left scalar multiplication of A° by any $\mathfrak{q} \in \mathbb{H}$ is defined as a map $\mathfrak{q}A^\circ : \mathfrak{P} \longrightarrow \mathfrak{P}$ by the setting

$$(\mathfrak{q}A^\circ)\{[\phi_n]\} := \{[\mathfrak{q}(A\phi_n)]\}, \tag{4.5}$$

for all $\{[\phi_n]\} \in \mathfrak{P}$. It is straightforward that $\mathfrak{q}A^\circ$ is a right linear operator. We also have the following properties for the operators:

Proposition 4.2. *For $A,B \in B(V_{\mathbb{H}}^R)$ and $\mathfrak{q} \in \mathbb{H}$, we have*

(a) $(A + B)^\circ = A^\circ + B^\circ$,
(b) $(\mathfrak{q}A)^\circ = \mathfrak{q}A^\circ$,
(c) $(AB)^\circ = A^\circ B^\circ$,
(d) $(A^\dagger)^\circ = (A^\circ)^\dagger$,
(e) ${\mathbb{I}_{V_{\mathbb{H}}^R}}^\circ = \mathbb{I}_{V_{\mathbb{H}}^R}$
(f) $\|A^\circ\|_\circ = \|A\|$.

Proof. Proofs of (a), (c) and (e) are straightforward from the definition of A°. Assertion (b) immediately follows from the (definition) Eq. 4.5 as follows: for any $\{[\phi_n]\} \in \mathfrak{P}$,

$$(\mathfrak{q}A^\circ)\{[\phi_n]\} = \{[\mathfrak{q}(A\phi_n)]\} = \{[(\mathfrak{q}A)\phi_n]\} = (\mathfrak{q}A)^\circ\{[\phi_n]\}.$$

To verify (d), let $C = (A^\circ)^\dagger$ and $u = \{[\phi_n]\}$ and $v = \{[\psi_n]\}$. Then

$$\langle A^\circ u \mid v\rangle_{\mathfrak{P}} = \langle u \mid Cv\rangle_{\mathfrak{P}}.$$

This implies that

$$\begin{aligned}\langle u \mid Cv\rangle_{\mathfrak{P}} &= \langle A^\circ u \mid v\rangle_{\mathfrak{P}} = \text{glim}(\langle A\phi_n|\psi_n\rangle_{V_{\mathbb{H}}^R}) = \text{glim}(\langle \phi_n|A^\dagger\psi_n\rangle_{V_{\mathbb{H}}^R})\\ &= \langle u \mid (A^\dagger)^\circ v\rangle_{\mathfrak{P}}.\end{aligned}$$

Therefore $(A^\dagger)^\circ = C = (A^\circ)^\dagger$, and this completes the proof of (d). Finally let us establish the equality $\|A^\circ\|_\circ = \|A\|$. Firstly note that for any $\phi \in V_{\mathbb{H}}^R$, from the Eq. 4.2, we have $\|\phi'\|_{\mathfrak{P}} = \|\phi\|_{V_{\mathbb{H}}^R}$. Now since $A^\circ\phi' = (A\phi)'$, for all $\phi \in V_{\mathbb{H}}^R$,

$$\|A^\circ\|_\circ = \sup_{\|\phi'\|_{\mathfrak{P}}=1} \|A^\circ\phi'\|_{\mathfrak{P}} = \sup_{\|\phi\|_{V_{\mathbb{H}}^R}=1} \|(A\phi)'\|_{\mathfrak{P}} = \sup_{\|\phi\|_{V_{\mathbb{H}}^R}=1} \|A\phi\|_{V_{\mathbb{H}}^R} = \|A\|.$$

Therefore the assertion (f) follows. □

The continuous right linear mapping A° extends to a unique right linear operator in $\mathfrak{H}$, which we also denote A°. Also in the Theorem 3.20, $B(V_{\mathbb{H}}^R)$ with left multiplication is a C^*-algebra with unity $\mathbb{I}_{V_{\mathbb{H}}^R}$. In the same manner, $B(\mathfrak{H})$ with left multiplication is a C^*-algebra with the same unity $\mathbb{I}_{V_{\mathbb{H}}^R}$. Also note that, no matter which Hilbert basis we choose to define a left multiplication the spaces $B(V_{\mathbb{H}}^R)$ and $B(\mathfrak{H})$ becomes C^*-algebras, and hence the results provided in this note are independent of the basis chosen.

Theorem 4.3. *The mapping*

$$B(V_{\mathbb{H}}^R) \longrightarrow B(\mathfrak{H}) \quad by \quad A \mapsto A^\circ$$

is a faithful $*$-representation.

Proof. The assertion (c) of Proposition 4.2 verifies that the above map is a homomorphism. To check the injectivity of this map, suppose that $A, B \in B(V_{\mathbb{H}}^R)$ with $A^\circ = B^\circ$. Then for any $\{[\phi_n]\} \in \mathfrak{P}$, we have

$$\begin{aligned}\{[A\phi_n]\} = \{[B\phi_n]\} &\Rightarrow \{(A-B)\phi_n\} \in \mathfrak{N}\\ &\Rightarrow \text{glim}(\langle (A-B)\phi_n \mid (A-B)\phi_n\rangle_{V_{\mathbb{H}}^R}) = 0.\end{aligned}$$

Let $\phi \in V_{\mathbb{H}}^R$, and choose $\phi_n = \phi$, $\forall n \in \mathbb{N}$. Then $\|(A-B)\phi\|_{V_{\mathbb{H}}^R} = 0$. This concludes that $A = B$. Therefore the above map is injective. Hence the theorem follows. □

Suppose $A \geq 0$, that is $\langle A\phi|\phi\rangle_{V_{\mathbb{H}}^R} \geq 0$ for all $\phi \in V_{\mathbb{H}}^R$. If $u = \{\phi_n\}' \in \mathfrak{P}$, then $\langle A\phi_n|\phi_n\rangle_{V_{\mathbb{H}}^R} \geq 0$ for all n, hence

$$\langle A^\circ u|u\rangle_{\mathfrak{P}} = \text{glim}\langle A\phi_n|\phi_n\rangle_{V_{\mathbb{H}}^R} \geq 0.$$

Hence $\langle A^\circ v|v\rangle_{\mathfrak{P}} \geq 0$ for all $v \in \mathfrak{H}$. Thus clearly for an operator A in $V_{\mathbb{H}}^R$ we have

$$A \geq 0 \Leftrightarrow A^\circ \geq 0. \tag{4.6}$$

Proposition 4.4. *If $A \in B(V_{\mathbb{H}}^R)$, then $\sigma_{ap}^S(A^\circ) = \sigma_{ap}^S(A)$.*

Proof. Let $\mathfrak{q} \in \mathbb{H}$. Then, $\mathfrak{q} \notin \sigma_{ap}^S(A)$ if and only if there exists $\epsilon > 0$ such that $R_{\mathfrak{q}}(A^\dagger)R_{\mathfrak{q}}(A) \geq \epsilon \mathbb{I}_{V_{\mathbb{H}}^R}$. By Eq. 4.6, this condition is equivalent to $R_{\mathfrak{q}}((A^\circ)^\dagger)R_{\mathfrak{q}}(A^\circ) \geq \epsilon \mathbb{I}_{V_{\mathbb{H}}^R}$ thus $\mathfrak{q} \notin \sigma_{ap}^S(A^\circ)$. □

The following theorem is the key result of Berberian extension.

Theorem 4.5. *For every operator $A \in B(V_{\mathbb{H}}^R)$, we have $\sigma_{ap}^S(A) = \sigma_{ap}^S(A^\circ) = \sigma_{pS}(A^\circ)$.*

Proof. From Propositions 3.15, 4.4 the relation $\sigma_{ap}^S(A) = \sigma_{ap}^S(A^\circ) \supseteq \sigma_{pS}(A^\circ)$ is clear. Let $\mathfrak{q} \in \sigma_{ap}^S(A)$. Then there exists a sequence $\{\phi_n\} \subseteq V_{\mathbb{H}}^R$ with $\|\phi_n\|_{V_{\mathbb{H}}^R} = 1$ such that $\|R_{\mathfrak{q}}(A)\phi_n\|_{V_{\mathbb{H}}^R} \to 0$. Set $u = \{\phi_n\}'$, clearly $\|u\|_{\mathfrak{P}} = 1$. Also

$$\|R_{\mathfrak{q}}(A^\circ)u\|_{\mathfrak{P}} = \text{glim}\|R_{\mathfrak{q}}(A)\phi_n\|_{V_{\mathbb{H}}^R} \to 0.$$

Therefore, by Proposition 3.13, $\mathfrak{q}$ is a right eigenvalue of A°. Hence $\mathfrak{q} \in \sigma_{pS}(A^\circ)$, which completes the proof. □

5. Application to Commutators in the Quaternionic Setting

In the complex setting, the Berberian extension is very useful in studying spectral properties of commutators [11]. Following the complex formalism, in this section, we shall study some properties of S-spectrum of commutators in the quaternionic setting.

Proposition 5.1. *Let $A, B \in B(V_{\mathbb{H}}^R)$ such that $AB = BA$, then*
(a) $\sigma_{ap}^S(A+B) \subseteq \sigma_{ap}^S(A) + \sigma_{ap}^S(B)$,
(b) $\sigma_{su}^S(A+B) \subseteq \sigma_{su}^S(A) + \sigma_{su}^S(B)$,
(c) $\sigma_S(A+B) \subseteq \sigma_S(A) + \sigma_S(B)$.

Proof. (a) Since $AB = BA$ we have $A^\circ B^\circ = B^\circ A^\circ$. Let $\mathfrak{q} \in \sigma_{ap}^S(A+B) = \sigma_{pS}(A^\circ + B^\circ)$. Let $Z = \ker(R_{\mathfrak{q}}(A^\circ + B^\circ))$. then $Z \neq \emptyset$. Let $\psi \in A^\circ Z$, then $\psi = A^\circ \phi$ for some $\phi \in Z$ and also $R_{\mathfrak{q}}(A^\circ + B^\circ)\phi = 0$. Now

$$R_{\mathfrak{q}}(A^\circ + B^\circ)\psi = R_{\mathfrak{q}}(A^\circ + B^\circ)A^\circ\phi = A^\circ R_{\mathfrak{q}}(A^\circ + B^\circ)\phi = 0.$$

Therefore $\psi \in Z$, hence $A^\circ Z \subseteq Z$. That is, Z is invariant under A°, and therefore $\sigma_{ap}^S(A^\circ|Z) \neq \emptyset$. Let $\mathfrak{p} \in \sigma_{ap}^S(A^\circ|Z) = \sigma_{pS}(A^\circ|Z)$, hence $A^\circ - \mathfrak{p}\mathbb{I}_{\mathfrak{H}} = 0$. Since $\mathfrak{q} \in \sigma_{pS}(A^\circ + B^\circ)$, we have $A^\circ + B^\circ - \mathfrak{q}\mathbb{I}_{\mathfrak{H}} = 0$, that is $B^\circ = \mathfrak{q}\mathbb{I}_{\mathfrak{H}} - A^\circ$.

Therefore,

$$B^\circ - (\mathfrak{q} - \mathfrak{p})\mathbb{I}_{\mathfrak{H}} = -(A^\circ - \mathfrak{p})\mathbb{I}_{\mathfrak{H}} = 0 \text{ on } Z.$$

Thus $\mathfrak{q} - \mathfrak{p} \in \sigma_{pS}(B^\circ|Z)$. Hence, from Proposition 4.4,

$$\begin{aligned} \mathfrak{q} &= \mathfrak{p} + (\mathfrak{q} - \mathfrak{p}) \in \sigma_{pS}(A^\circ) + \sigma_{pS}(B^\circ) = \sigma_{ap}^S(A^\circ) + \sigma_{ap}^S(B^\circ) \\ &= \sigma_{ap}^S(A) + \sigma_{ap}^S(B). \end{aligned}$$

This completes the proof of (a).
(b) Since $AB = BA$, we have $A^\dagger B^\dagger = B^\dagger A^\dagger$, and therefore (a) holds for $A^\dagger, B^\dagger$. Further from Proposition 3.18, part (b), $\sigma_{su}^S(A) = \sigma_{ap}^S(A^\dagger)$. Thus (b) follows.

(c) For any $A \in B(V_{\mathbb{H}}^R)$, from Eq. 3.8, Proposition 3.15, we have $\sigma_S(A) = \sigma_{pS}(A) \cup \sigma_{su}^S(A) \subseteq \sigma_{ap}^S(A) \cup \sigma_{su}^S(A)$. And clearly $\sigma_{ap}^S(A), \sigma_{su}^S(A) \subseteq \sigma_S(A)$. Therefore, from (a) and (b), we have

$$\sigma_{ap}^S(A+B) \subseteq \sigma_{ap}^S(A) + \sigma_{ap}^S(B) \subseteq \sigma_S(A) + \sigma_S(B)$$

and

$$\sigma_{su}^S(A+B) \subseteq \sigma_{su}^S(A) + \sigma_{su}^S(B) \subseteq \sigma_S(A) + \sigma_S(B).$$

Thus

$$\sigma_S(A+B) \subseteq \sigma_{ap}^S(A+B) \cup \sigma_{su}^S(A+B) \subseteq \sigma_S(A) + \sigma_S(B).$$

Hence the inclusion (c) holds. □

Definition 5.2. Given $S, T \in B(V_{\mathbb{H}}^R)$, the commutator $C(S,T) : B(V_{\mathbb{H}}^R) \longrightarrow B(V_{\mathbb{H}}^R)$ is the mapping

$$C(S,T)(A) = SA - AT = \mathbf{L}_S(A) - \mathbf{R}_T(A), \quad \text{for all } A \in B(V_{\mathbb{H}}^R),$$

where $\mathbf{L}_S(A) = SA$ and $\mathbf{R}_T(A) = AT$. It is clear that $A \in B(V_{\mathbb{H}}^R)$ intertwines the pair (S,T) precisely when $C(S,T) = 0$.

Remark 5.3. It is worth noting the following results: for any $\mathfrak{q} \in \mathbb{H}$ and $S, T \in B(V_{\mathbb{H}}^R)$,

(1) $\mathbf{L}_S\mathbf{R}_T = \mathbf{R}_T\mathbf{L}_S$,
(2) $R_{\mathfrak{q}}(\mathbf{L}_S) = R_{\mathfrak{q}}(S)$,
(3) $R_{\mathfrak{q}}(\mathbf{R}_T)A = AR_{\mathfrak{q}}(T)$; for all $A \in B(V_{\mathbb{H}}^R)$,
(4) $\mathbf{L}_{R_{\mathfrak{q}}(S)} = R_{\mathfrak{q}}(\mathbf{L}_S)$,
(5) $\mathbf{R}_{R_{\mathfrak{q}}(T)} = R_{\mathfrak{q}}(\mathbf{R}_T)$.

The verifications of these results are elementary.

The next proposition gathers some useful identities to prove the S-spectral properties of commutators which are provided in Theorem 5.5.

Proposition 5.4. *For arbitrary operators $S, T \in B(V_{\mathbb{H}}^R)$ the following assertions hold true:*

(a) $\sigma_{ap}^S(\mathbf{L}_S) = \sigma_{ap}^S(S)$,
(b) $\sigma_{ap}^S(\mathbf{R}_T) = \sigma_{su}^S(T)$,
(c) $\sigma_{su}^S(\mathbf{L}_S) = \sigma_{su}^S(S)$,
(d) $\sigma_{su}^S(\mathbf{R}_T) = \sigma_{ap}^S(T)$.

Proof. (a) To prove (a), let $\mathfrak{q} \in \sigma_{ap}^S(\mathbf{L}_S)$, then there exists a sequence $\{A_n\} \subseteq B(V_{\mathbb{H}}^R)$ with $\|A_n\| = 1$ such that $\|R_{\mathfrak{q}}(\mathbf{L}_S)A_n\| \to 0$. That is, $\|R_{\mathfrak{q}}(S)A_n\| \to 0$. Set $\psi_n = \dfrac{\phi}{\|A_n\phi\|_{V_{\mathbb{H}}^R}}$, for all n and for some $0 \neq \phi \in V_{\mathbb{H}}^R$. Then $\|A_n\psi_n\|_{V_{\mathbb{H}}^R} = 1$, for all n and $\|R_{\mathfrak{q}}(S)A_n\psi_n\|_{V_{\mathbb{H}}^R} \to 0$. Thus $\mathfrak{q} \in \sigma_{ap}^S(S)$ and $\sigma_{ap}^S(\mathbf{L}_S) \subseteq \sigma_{ap}^S(S)$. To see the other inclusion, let $\mathfrak{q} \in \sigma_{ap}^S(S)$, then there exists a sequence $\{\psi_n\} \subseteq V_{\mathbb{H}}^R$ with $\|\psi_n\|_{V_{\mathbb{H}}^R} = 1$ and $\|R_{\mathfrak{q}}(S)\psi_n\|_{V_{\mathbb{H}}^R} \to 0$. Pick a linear functional Φ in $(V_{\mathbb{H}}^R)^*$ which is the dual of $V_{\mathbb{H}}^R$ with $\|\Phi\|^* = 1$, where $\|\cdot\|^*$ is a norm on the dual of $V_{\mathbb{H}}^R$. Define, for each n, an operator $A_n \in B(V_{\mathbb{H}}^R)$ by

$$A_n\phi = \psi_n\Phi(\phi), \quad \text{for all } \phi \in V_{\mathbb{H}}^R.$$

Then $\|A_n\| = 1$ for all n, and

$$\|R_{\mathfrak{q}}(\mathbf{L}_S)A_n\| = \|R_{\mathfrak{q}}(S)A_n\| = \|R_{\mathfrak{q}}(S)\psi_n\|_{V_{\mathbb{H}}^R} \to 0.$$

Thus $\mathfrak{q} \in \sigma_{ap}^S(\mathbf{L}_S)$ and therefore $\sigma_{ap}^S(\mathbf{L}_S) = \sigma_{ap}^S(S)$.

(b) To establish the equality $\sigma_{ap}^S(\mathbf{R}_T) = \sigma_{su}^S(T)$, let $\mathfrak{q} \notin \sigma_{su}^S(T)$, that is $R_{\mathfrak{q}}(T)$ is surjective. Now for each $A \in B(V_{\mathbb{H}}^R)$,

$$\|R_{\mathfrak{q}}(\mathbf{R}_T)A\| = \|AR_{\mathfrak{q}}(T)\| \geq \|AR_{\mathfrak{q}}(T)\phi\|_{V_{\mathbb{H}}^R}, \quad \text{for all } \phi \in V_{\mathbb{H}}^R.$$

That is, as $R_{\mathfrak{q}}(T)$ is surjective, $\|R_{\mathfrak{q}}(\mathbf{R}_T)A\| \geq \|A\psi\|_{V_{\mathbb{H}}^R}$, for all $\psi = R_{\mathfrak{q}}(T)\phi \in V_{\mathbb{H}}^R$. Hence

$$\|R_{\mathfrak{q}}(\mathbf{R}_T)A\| \geq \sup_{\|\psi\|=1} \|A\psi\|_{V_{\mathbb{H}}^R} = \|A\|, \quad \text{for all } A \in B(V_{\mathbb{H}}^R).$$

Therefore $R_{\mathfrak{q}}(\mathbf{R}_T)$ is bounded below, and hence by Proposition 3.19, $\mathfrak{q} \notin \sigma_{ap}^S(\mathbf{R}_T)$. Conversely suppose that $R_{\mathfrak{q}}(\mathbf{R}_T)$ is bounded below. Then there exists $c > 0$ such that $c\|A\| \leq |||R_{\mathfrak{q}}(\mathbf{R}_T)||| = \|AR_{\mathfrak{q}}(T)\|$, for all $A \in B(V_{\mathbb{H}}^R)$; where $|||\cdot|||$ is the norm on $B(B(V_{\mathbb{H}}^R))$. Choose a unit vector $\psi \in V_{\mathbb{H}}^R$. For arbitrary linear functional $\Phi \in (V_{\mathbb{H}}^R)^*$, let $A_\Phi \in B(V_{\mathbb{H}}^R)$ given by

$$A_\Phi(\phi) = \psi\Phi(\phi), \quad \text{for all } \phi \in V_{\mathbb{H}}^R.$$

Then

$$c\|\Phi\|^* = c\|A_\Phi\| \leq \|A_\Phi R_{\mathfrak{q}}(T)\| = \|\Phi \circ R_{\mathfrak{q}}(T)\|, \quad \text{for all } \Phi \in (V_{\mathbb{H}}^R)^*.$$

Hence $R_{\mathfrak{q}}(T)^\dagger$ is bounded below. That is, by Proposition 3.4, $R_{\mathfrak{q}}(T)^\dagger$ is injective. Therefore by Propositions 3.3, $\text{ran}(R_{\mathfrak{q}}(T))^\perp = \ker(R_{\mathfrak{q}}(T)^\dagger) = \{0\}$, and so $R_{\mathfrak{q}}(T)$ is surjective. Thus we have $\sigma_{ap}^S(\mathbf{R}_T) = \sigma_{su}^S(T)$.

(c) Now

$$\begin{aligned} \mathfrak{q} \notin \sigma_{su}^S(\mathbf{L}_S) &\iff R_{\mathfrak{q}}(\mathbf{L}_S) \text{is sujective} \iff R_{\mathfrak{q}}(S) \text{ is sujective} \\ &\iff \mathfrak{q} \notin \sigma_{su}^S(S). \end{aligned}$$

Therefore $\sigma_{su}^S(\mathbf{L}_S) = \sigma_{su}^S(S)$.

(d) In order to verify the inclusion $\sigma_{su}^S(\mathbf{R}_T) \subseteq \sigma_{ap}^S(T)$, let $\mathfrak{q} \notin \sigma_{ap}^S(T)$, then by Proposition 3.19, $R_{\mathfrak{q}}(T)$ is bounded below on $V_{\mathbb{H}}^R$. Therefore, by Proposition 3.4, $R_{\mathfrak{q}}(T)$ is injective. Thus by Proposition 3.7, $R_{\mathfrak{q}}(T)$ is left invertible on $V_{\mathbb{H}}^R$. Therefore, there exist $P \in B(V_{\mathbb{H}}^R)$ such that $PR_{\mathfrak{q}}(T) = \mathbb{I}_{V_{\mathbb{H}}^R}$. This implies that

$$R_{\mathfrak{q}}(\mathbf{R}_T)\mathbf{R}_P A = \mathbf{R}_P(A)R_{\mathfrak{q}}(T) = APR_{\mathfrak{q}}(T) = A, \quad \text{for all } A \in B(V_{\mathbb{H}}^R).$$

That is, $R_{\mathfrak{q}}(\mathbf{R}_T)$ is right invertible on $B(V_{\mathbb{H}}^R)$, thus by Proposition 3.6, $R_{\mathfrak{q}}(\mathbf{R}_T)$ is surjective. Hence $\mathfrak{q} \notin \sigma_{su}^S(\mathbf{R}_T)$, and we get $\sigma_{su}^S(\mathbf{R}_T) \subseteq \sigma_{ap}^S(T)$. To verify the other inclusion $\sigma_{ap}^S(T) \subseteq \sigma_{su}^S(\mathbf{R}_T)$, suppose that $\mathfrak{q} \notin \sigma_{su}^S(\mathbf{R}_T)$, then $R_{\mathfrak{q}}(\mathbf{R}_T)$ is surjective. This implies that for each $A \in B(V_{\mathbb{H}}^R)$, there exists $B \in B(V_{\mathbb{H}}^R)$ such that $R_{\mathfrak{q}}(\mathbf{R}_T)B = A$. That is, $BR_{\mathfrak{q}}(T) = A$. Assuming $A, B \neq 0$ without loss of generality, we get

$$\|R_{\mathfrak{q}}(T)\| \geq \frac{\|A\|}{\|B\|}.$$

This gives that $R_{\mathfrak{q}}(T)$ is bounded below, and $\mathfrak{q} \notin \sigma_{ap}^S(T)$. Therefore the equality $\sigma_{su}^S(\mathbf{R}_T) = \sigma_{ap}^S(T)$ holds. $\square$

The following theorem is the main result about the S-spectral properties of commutators which we provide in this note.

Theorem 5.5. *For arbitrary operators $S, T \in B(V_{\mathbb{H}}^R)$ the following assertions hold true for their commutator $C(S,T)$.*

(a) $\sigma_S(C(S,T)) = \sigma_S(S) - \sigma_S(T)$,
(b) $\sigma_{ap}^S(C(S,T)) = \sigma_{ap}^S(S) - \sigma_{su}^S(T)$,
(c) $\sigma_{su}^S(C(S,T)) = \sigma_{su}^S(S) - \sigma_{ap}^S(T)$.

Proof. To prove (b), In order to establish $\sigma_{ap}^S(S) - \sigma_{su}^S(T) \subseteq \sigma_{ap}^S(C(S,T))$, let $\mathfrak{q} \in \sigma_{ap}^S(S)$ and $\mathfrak{p} \in \sigma_{su}^S(T)$. It follows, from (a) and (b) in the Proposition 5.4, that $\mathfrak{q} \in \sigma_{ap}^S(\mathbf{L}_S)$ and $\mathfrak{p} \in \sigma_{ap}^S(\mathbf{R}_T)$. By Proposition 4.5 we have

$$\mathfrak{q} \in \sigma_{ap}^S(\mathbf{L}_S) = \sigma_{ap}^S(\mathbf{L}_S^\circ) = \sigma_{pS}(\mathbf{L}_S^\circ).$$

Therefore, there exists $A \in B(V_{\mathbb{H}}^R)$ such that $A \neq 0$ and $(\mathbf{L}_S^\circ - \mathfrak{q})A = 0$. That is, $(S^\circ - \mathfrak{q})A = 0$. Again by Proposition 4.5,

$$\mathfrak{p} \in \sigma_{ap}^S(\mathbf{R}_T) = \sigma_{ap}^S(\mathbf{R}_T^\circ) = \sigma_{pS}(\mathbf{R}_T^\circ).$$

Therefore there exists $B \in B(V_{\mathbb{H}}^R)$ such that $B \neq 0$ and $(\mathbf{R}_T^\circ - \mathfrak{p})B = 0$. That is $B(T^\circ - \mathfrak{p}) = 0$. Consider

$$\begin{aligned}(C(S,T)^\circ - \mathfrak{q} + \mathfrak{p})AB &= (\mathbf{L}_S^\circ - \mathbf{R}_T^\circ - \mathfrak{q} + \mathfrak{p})AB \\ &= (\mathbf{L}_S^\circ - \mathfrak{q})AB - (\mathbf{R}_T^\circ - \mathfrak{p})AB \\ &= (S^\circ - \mathfrak{q})AB - AB(T^\circ - \mathfrak{p}) = 0.\end{aligned}$$

Thus

$$\mathfrak{q} - \mathfrak{p} \in \sigma_{pS}(C(S,T)^\circ) = \sigma_{ap}^S(C(S,T)^\circ) = \sigma_{ap}^S(C(S,T)).$$

Therefore $\sigma_{ap}^S(S) - \sigma_{su}^S(T) \subseteq \sigma_{ap}^S(C(S,T))$. Now since $\sigma_{ap}^S(\mathbf{L}_S) = \sigma_{ap}^S(S)$ and $\sigma_{ap}^S(\mathbf{R}_T) = \sigma_{su}^S(T)$, we get from part (a) of Proposition 5.1,

$$\sigma_{ap}^S(C(S,T)) = \sigma_{ap}^S(\mathbf{L}_S - \mathbf{R}_T) \subseteq \sigma_{ap}^S(\mathbf{L}_S) - \sigma_{ap}^S(\mathbf{R}_T) \subseteq \sigma_{ap}^S(S) - \sigma_{su}^S(T).$$

This concludes the proof for (b).
(c) Applying $\sigma_{su}^S(\mathbf{L}_S) = \sigma_{su}^S(S)$, and $\sigma_{su}^S(\mathbf{R}_T) = \sigma_{ap}^S(T)$ in part (b) of Proposition 5.1, the inclusion $\subseteq$ in assertion (c) is established. Next to show that $\sigma_{su}^S(S) - \sigma_{ap}^S(T) \subseteq \sigma_{su}^S(C(S,T))$, let $\mathfrak{q} \in \sigma_{su}^S(S)$ and $\mathfrak{p} \in \sigma_{ap}^S(T)$. It follows from (a) and (b) in Proposition 5.4, and Proposition 4.5 that

$$\begin{aligned}\mathfrak{q} \in \sigma_{ap}^S(\mathbf{R}_S) &= \sigma_{ap}^S(\mathbf{R}_S^\circ) = \sigma_{pS}(\mathbf{R}_S^\circ) \quad \text{and} \quad \mathfrak{p} \in \sigma_{ap}^S(\mathbf{L}_T) \\ &= \sigma_{ap}^S(\mathbf{L}_T^\circ) = \sigma_{pS}(\mathbf{L}_T^\circ).\end{aligned}$$

Thus by the definition of point spectrum $\overline{\mathfrak{q}} \in \sigma_{pS}(\mathbf{R}_S^\circ)$ and $\overline{\mathfrak{p}} \in \sigma_{pS}(\mathbf{L}_T^\circ)$. Therefore, there exists $A, B \in B(V_{\mathbb{H}}^R)$ with $A \neq 0$ and $B \neq 0$ such that $A(S^\circ - \overline{\mathfrak{q}}) = 0$ and $(T^\circ - \mathfrak{p})B = 0$. Hence, by Proposition 3.2, we have $((S^\circ)^\dagger - \mathfrak{q})A^\dagger = 0$ and $B^\dagger((T^\circ)^\dagger - \mathfrak{p}) = 0$. Hence, by Proposition 3.2,

$$\begin{aligned}((C(S,T)^{\circ})^{\dagger}-\mathfrak{q}+\mathfrak{p})A^{\dagger}B^{\dagger} &= ((\mathbf{L}_S^{\circ})^{\dagger}-\mathfrak{q})A^{\dagger}B^{\dagger}-((\mathbf{R}_T^{\circ})^{\dagger}-\mathfrak{p})A^{\dagger}B^{\dagger}\\ &= ((S^{\circ})^{\dagger}-\mathfrak{q})A^{\dagger}B^{\dagger}-A^{\dagger}B^{\dagger}((T^{\circ})^{\dagger}-\mathfrak{p})=0.\end{aligned}$$

Therefore $\mathfrak{q}-\mathfrak{p}\in\sigma_{pS}((C(S,T)^{\circ})^{\dagger})$. By proposition 4.2, $\mathfrak{q}-\mathfrak{p}\in\sigma_{pS}((C(S,T)^{\circ})^{\dagger})=\sigma_{pS}((C(S,T)^{\dagger})^{\circ})$. Now by propositions 4.5 and 3.18, we have

$$\mathfrak{q}-\mathfrak{p}\in\sigma_{pS}((C(S,T)^{\dagger})^{\circ})=\sigma_{ap}^S(C(S,T)^{\dagger})=\sigma_{su}^S(C(S,T)).$$

Hence $\sigma_{su}^S(S)-\sigma_{ap}^S(T)\subseteq\sigma_{su}^S(C(S,T))$, which completes the proof of (c). To establish (a), let $S,T\in B(V_{\mathbb{H}}^R)$. Since $\mathbf{L}_S\mathbf{R}_T(A)=\mathbf{R}_T\mathbf{L}_S(A)=SAT$ for all $A\in B(V_{\mathbb{H}}^R)$, $\mathbf{L}_S$,$\mathbf{R}_T\in B(B(V_{\mathbb{H}}^R))$ commute. Let $\mathfrak{q}\in\sigma_S(\mathbf{L}_S)$, then if $\ker(\mathbf{L}_S)\neq\{0\}$, there exists $A\in\mathcal{B}(V_{\mathbb{H}}^R)$ such that $A\neq 0$ and $R_{\mathfrak{q}}(\mathbf{L}_S)(A)=0$. That is

$$S^2A-2\text{Re}(\mathfrak{q})SA+|\mathfrak{q}|^2A=(S^2-2\text{Re}(\mathfrak{q})S+|\mathfrak{q}|^2)A=0.$$

Hence $(S^2-2\text{Re}(\mathfrak{q})S+|\mathfrak{q}|^2)A\phi=0$, for some $\phi\in V_{\mathbb{H}}^R$ as $A\neq 0$, and therefore $\ker(R_{\mathfrak{q}}(S))\neq\{0\}$. If $\text{ran}(R_{\mathfrak{q}}(\mathbf{L}_S))\neq B(V_{\mathbb{H}}^R)$, then there exists $B\in B(V_{\mathbb{H}}^R)$ such that $R_{\mathfrak{q}}(\mathbf{L}_S)(A)\neq B$ for all $A\in B(V_{\mathbb{H}}^R)$. That is, $S^2A-2\text{Re}(\mathfrak{q})SA+|\mathfrak{q}|^2A\neq B$ for all $A\in B(V_{\mathbb{H}}^R)$. In other words, $R_{\mathfrak{q}}(S)A\phi\neq B\phi$ for all $A\in B(V_{\mathbb{H}}^R)$ and $\phi\in V_{\mathbb{H}}^R$). Hence $\text{ran}(R_{\mathfrak{q}}(S))\neq V_{\mathbb{H}}^R$. As a conclusion $\mathfrak{q}\in\sigma_S(S)$ and hence $\sigma_S(\mathbf{L}_S)\subseteq\sigma_S(S)$.

Now let $\mathfrak{q}\in\sigma_S(\mathbf{R}_T)$. If $\ker(R_{\mathfrak{q}}(\mathbf{R}_T))\neq\{0\}$, then there exists $A\in B(V_{\mathbb{H}}^R)$ such that $A\neq 0$ and $R_{\mathfrak{q}}(\mathbf{R}_T)(A)=0$, that is $AR_{\mathfrak{q}}(T)=0$. Thus $R_{\mathfrak{q}}(T)\phi=0$ for some $0\neq\phi\in V_{\mathbb{H}}^R$, and therefore $\ker(R_{\mathfrak{q}}(T))\neq\{0\}$. If $\text{ran}(R_{\mathfrak{q}}(\mathbf{R}_T))^{\perp}\neq B(V_{\mathbb{H}}^R)$, then there exists $B\in\mathcal{B}(V_{\mathbb{H}}^R)$ such that $R_{\mathfrak{q}}(\mathbf{R}_T)(A)\neq B$, for all $A\in B(V_{\mathbb{H}}^R)$. That is $AR_{\mathfrak{q}}(T)\neq B$, for all $A\in B(V_{\mathbb{H}}^R)$, and hence $\mathbb{I}_{V_{\mathbb{H}}^R}R_{\mathfrak{q}}(T)\neq B$. Therefore $\text{ran}(R_{\mathfrak{q}}(T))\neq V_{\mathbb{H}}^R$. Hence we can conclude that $\mathfrak{q}\in\sigma_S(T)$ and $\sigma_S(\mathbf{R}_T)\subseteq\sigma_S(T)$. Because $C(S,T)=\mathbf{L}_S-\mathbf{R}_T$, by part (c) of Proposition 5.1 we have

$$\sigma_S(C(S,T))\subseteq\sigma_S(\mathbf{L}_S)-\sigma_S(\mathbf{R}_T)\subseteq\sigma_S(S)-\sigma_S(T).$$

This establishes the inclusion $\subseteq$ in assertion (a). Now for each $A,B\in B(V_{\mathbb{H}}^R)$, we have

$$C(\mathbf{L}_S,\mathbf{R}_T)AB=\mathbf{L}_SAB-A\mathbf{R}_TB=SAB-ABT=C(S,T)AB.$$

This implies that

$$C(\mathbf{L}_S,\mathbf{R}_T)=C(S,T),\ \forall\, S,T\in B(V_{\mathbb{H}}^R). \tag{5.1}$$

On the other hand, using Eq. 5.1, from part (c),

$$\sigma_{su}^S(C(S,T))=\sigma_{su}^S(C(\mathbf{L}_S,\mathbf{R}_T))\supseteq\sigma_{su}^S(\mathbf{L}_S)-\sigma_{ap}^S(\mathbf{R}_T)=\sigma_{su}^S(S)-\sigma_{su}^S(T) \tag{5.2}$$

as $\sigma_{ap}^S(\mathbf{R}_T)=\sigma_{su}^S(T)$ and $\sigma_{su}^S(\mathbf{L}_S)=\sigma_{su}^S(S)$. Similarly from part (b), we get

$$\sigma_{ap}^S(C(S,T))=\sigma_{ap}^S(C(\mathbf{L}_S,\mathbf{R}_T))\supseteq\sigma_{ap}^S(\mathbf{L}_S)-\sigma_{su}^S(\mathbf{R}_T)=\sigma_{ap}^S(S)-\sigma_{ap}^S(T) \tag{5.3}$$

as $\sigma_{ap}^S(\mathbf{L}_S)=\sigma_{ap}^S(S)$ and $\sigma_{su}^S(\mathbf{R}_T)=\sigma_{ap}^S(T)$. Now the inclusions (5.2) and (5.3) guarantee that the other inclusion in assertion (a) holds,

$$\sigma_S(C(S,T)) \supseteq \sigma_S(\mathbf{L}_S) - \sigma_S(\mathbf{R}_T) \subseteq \sigma_S(S) - \sigma_S(T).$$

Therefore the assertion (a) follows. Hence the theorem holds. □

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