Stat Papers
DOI 10.1007/s00362-016-0856-6

REGULAR ARTICLE

Logistic Liu Estimator under stochastic linear restrictions

Nagarajah Varathan[1,2] · **Pushpakanthie Wijekoon**[3]

Received: 9 August 2016 / Revised: 8 November 2016

Abstract In order to overcome the problem of multicollinearity in logistic regression, several researchers proposed alternative estimators when exact linear restrictions are available in addition to sample model. However, in practical situations the linear restrictions are not always exact and mostly their nature is stochastic. In this paper, we propose a new estimator called stochastic restricted Liu maximum likelihood estimator (SRLMLE) by incorporating Liu estimator to the logistic regression model when the linear restrictions are stochastic. Moreover, the conditions for superiority of SRLMLE over the maximum likelihood estimator (MLE), stochastic restricted maximum likelihood estimator (SRMLE) and restricted Liu logistic estimator (RLLE) are derived with respect to mean square error criterion. Finally, the performance of the new estimator over MLE, LLE, SRMLE and RLLE is investigated in the sense of scalar mean squared error by conducting a Monte Carlo simulation and using a numerical example.

Keywords Logistic regression · Multicollinearity · Liu estimator · Stochastic restricted Liu maximum likelihood estimator · Stochastic linear restrictions

✉ Nagarajah Varathan
varathan10@gmail.com

Pushpakanthie Wijekoon
pushpaw@pdn.ac.lk

1 Postgraduate Institute of Science, University of Peradeniya, Peradeniya, Sri Lanka

2 Department of Mathematics and Statistics, University of Jaffna, Jaffna, Sri Lanka

3 Department of Statistics and Computer Science, University of Peradeniya, Peradeniya, Sri Lanka

Published online: 23 November 2016

1 Introduction

It is well known that the Maximum likelihood estimation technique is preferred to estimate the parameters of the logistic regression model. When the predictors in the logistic regression model exhibit high degree of correlations, which is called as multicollinearity, the variance of the maximum likelihood estimator (MLE) is inflated so that one cannot obtain efficient estimates. To overcome this issue in the logistic regression, several researchers proposed alternative biased estimators to MLE. To obtain these estimators researchers combined the existing biased estimators derived for the linear regression model. Three types of biased estimators available in the linear regression model are (i) biased estimators based only on sample information, (ii) biased estimators based on sample information and priori available exact linear restrictions on parameter space, and (iii) biased estimators based on sample information and stochastic linear restrictions on parameter space.

Some of the improved logistic regression estimators of the first kind are namely the ridge logistic estimator (RLE) (Schaefer et al. 1984), principal component logistic estimator (PCLE) (Aguilera et al. 2006), modified logistic ridge estimator (MLRE) (Nja et al. 2013), logistic Liu estimator (LLE) (Mansson et al. 2012), Liu-type estimator (Inan and Erdogan 2013) and almost unbiased ridge logistic estimator (AURLE) (Wu and Asar 2016). In the presence of exact prior information in addition to the sample information, several researchers proposed different estimators for the respective parameter β. Duffy and Santner (1989) introduced the restricted maximum likelihood estimator (RMLE) by incorporating the exact linear restriction on the unknown parameters. Şiray et al. (2015) proposed a new estimator called restricted logistic Liu estimator (RLLE) by replacing MLE by RMLE in the logistic Liu estimator. However, RLLE estimator did not satisfy the stated linear restrictions. Consequently, Wu (2015) proposed a modified restricted Liu estimator in logistic regression, which satisfies the linear restrictions. Later Wu and Asar (2015) investigated the theoretical results related to the mean squared error properties of the restricted estimator compared to MLE, RMLE and Liu estimator. When the restrictions on the parameters are stochastic, Nagarajah and Wijekoon (2015) introduced a new estimator called stochastic restricted maximum likelihood estimator (SRMLE), and derived the superiority conditions of SRMLE over the estimators logistic ridge estimator (LRE), logistic Liu estimator (LLE) and RMLE. Later the stochastic restricted ridge maximum likelihood estimator (SRRMLE) was proposed by Varathan and Wijekoon (2016) by incorporating ridge logistic estimator in the presence of stochastic restrictions.

In this paper, an estimator namely, Stochastic Restricted Liu Maximum Likelihood Estimator (SRLMLE) is proposed to estimate the parameters when the linear stochastic restrictions are available in addition to the sample information in the logistic regression model. The rest of the paper is organized as follows. The model specification and existing estimators are given in Sect. 2. The Stochastic Restricted Liu Maximum likelihood Estimator (SRLMLE) has been proposed and derived its asymptotic properties in Sect. 3. In Sect. 4, the mean square error matrix and the scalar mean square error for this new estimator are obtained. The theoretical performance of the proposed estimator over some existing estimators is derived in Sect. 5. The performance of the proposed estimator with respect to the Scalar Mean Squared Error (SMSE) is investigated by performing a Monte Carlo simulation study in Sect. 6. In Sect. 7, a numerical example is given to illustrate the theoretical findings of the proposed estimator. Finally, the conclusions of the study is presented in Sect. 8.

2 Model specification and existing estimators

Consider the general logistic regression model

$$y_i = \pi_i + \varepsilon_i, \quad i = 1, \ldots, n \tag{2.1}$$

which follows Bernoulli distribution with parameter π_i as

$$\pi_i = \frac{\exp(x_i'\beta)}{1 + \exp(x_i'\beta)}, \tag{2.2}$$

where x_i is the ith row of X, which is an $n \times (p+1)$ data matrix with p explanatory variables and β is a $(p+1) \times 1$ vector of coefficients, ε_i are independent with mean zero and variance $\pi_i(1-\pi_i)$ of the response y_i. The Maximum likelihood method is the well-known estimation technique to estimate the parameter β, and the maximum likelihood estimator (MLE) of β can be obtained as follows:

$$\hat{\beta}_{MLE} = C^{-1} X' \hat{W} Z, \tag{2.3}$$

where $C = X'\hat{W}X$; Z is the column vector with ith element equals $logit(\hat{\pi}_i) + \frac{y_i - \hat{\pi}_i}{\hat{\pi}_i(1-\hat{\pi}_i)}$ and $\hat{W} = diag[\hat{\pi}_i(1-\hat{\pi}_i)]$, which is an unbiased estimate of β. The covariance matrix of $\hat{\beta}_{MLE}$ is

$$Cov(\hat{\beta}_{MLE}) = \{X'\hat{W}X\}^{-1}. \tag{2.4}$$

By following Liu (1993), Urgan and Tez (2008) and Mansson et al. (2012), the Logistic Liu estimator (LLE) based on the sample information is defined as

$$\begin{aligned} \hat{\beta}_{LLE} &= (C+I)^{-1}(C+dI)\hat{\beta}_{MLE} \\ &= Z_d \hat{\beta}_{MLE}. \end{aligned} \tag{2.5}$$

where $Z_d = (C+I)^{-1}(C+dI)$ and $0 < d < 1$ is a parameter.

The asymptotic properties of LLE are

$$E[\hat{\beta}_{LLE}] = E[Z_d \hat{\beta}_{MLE}] = Z_d \beta, \tag{2.6}$$

$$\begin{aligned} Cov[\hat{\beta}_{LLE}] &= Cov[Z_d \hat{\beta}_{MLE}] \\ &= Z_d C^{-1} Z_d' \\ &= (C+I)^{-1}(C+dI)(I+dC^{-1})(C+I)^{-1}, \end{aligned} \tag{2.7}$$

and

$$Bias[\hat{\beta}_{LLE}] = E[Z_d \hat{\beta}_{MLE}] - \beta = (Z_d - I)\beta. \tag{2.8}$$

According to the literature, to improve the performance of the estimators, prior information is incorporated to the linear model either as exact linear restrictions or stochastic linear restrictions when multicollinearity exists. The resulted estimator is called Restricted estimator with exact restrictions, and the Mixed estimator (Theil and Goldberger 1961) under stochastic linear restrictions. In most of the comparisons it was noted that adding stochastic linear restrictions show higher performance than the exact case.

Suppose that the following stochastic linear prior information is given in addition to the general logistic regression model (2.1).

$$h = H\beta + \upsilon; \quad E(\upsilon) = \mathbf{0}, \quad Cov(\upsilon) = \Psi. \tag{2.9}$$

where h is an $(q \times 1)$ stochastic known vector, H is a $(q \times (p+1))$ of full rank $q \leq (p+1)$ known elements and υ is an $(q \times 1)$ random vector of disturbances with mean $\mathbf{0}$ and dispersion matrix Ψ, and Ψ is assumed to be known $(q \times q)$ positive definite matrix. Further, it is assumed that υ is stochastically independent of ε, i.e) $E(\varepsilon\upsilon') = 0$.

In the presence of exact linear restrictions on regression coefficients ($\upsilon = 0$ in (2.9)) in addition to the logistic regression model (2.1), Duffy and Santner (1989) proposed the following Restricted Maximum Likelihood Estimator (RMLE).

$$\hat{\beta}_{RMLE} = \hat{\beta}_{MLE} - C^{-1}H'(HC^{-1}H')^{-1}(H\hat{\beta}_{MLE} - h) \tag{2.10}$$

The asymptotic variance and bias of $\hat{\beta}_{RMLE}$ are

$$Cov(\hat{\beta}_{RMLE}) = C^{-1} - C^{-1}H'(HC^{-1}H')^{-1}HC^{-1} \tag{2.11}$$

and

$$Bias(\hat{\beta}_{RMLE}) = -C^{-1}H'(HC^{-1}H')^{-1}(H\beta - h) \quad \text{respectively.} \tag{2.12}$$

Following Duffy and Santner (1989), Şiray et al. (2015) proposed the following Restricted Logistic Liu estimator (RLLE) when exact linear restrictions are available in addition to sample model (2.1).

$$\begin{aligned} \hat{\beta}_{RLLE} &= (C+I)^{-1}(C+dI)\hat{\beta}_{RMLE} \\ &= Z_d \hat{\beta}_{RMLE}. \end{aligned} \tag{2.13}$$

The asymptotic variance and bias of $\hat{\beta}_{RLLE}$ are obtained as

$$Cov(\hat{\beta}_{RLLE}) = Z_d A Z_d', \tag{2.14}$$

and

$$\begin{aligned} Bias(\hat{\beta}_{RLLE}) &= (Z_d - I)\beta + Z_d\delta^* \\ &= \delta_3 \ (say), \quad \text{respectively,} \end{aligned} \tag{2.15}$$

where $A = C^{-1} - C^{-1}H'(HC^{-1}H')^{-1}HC^{-1}$ and $\delta^* = -C^{-1}H'(HC^{-1}H')^{-1}(H\beta - h)$. Note that A is the variance and δ^* is the bias of $\hat{\beta}_{RMLE}$.

When the linear restriction is stochastic as in (2.9) in addition to the logistic regression model (2.1), Nagarajah and Wijekoon (2015) proposed the following stochastic restricted maximum likelihood estimator (SRMLE).

$$\hat{\beta}_{SRMLE} = \hat{\beta}_{MLE} + C^{-1}H'(\Psi + HC^{-1}H')^{-1}(h - H\hat{\beta}_{MLE}) \tag{2.16}$$

The asymptotic properties of SRMLE are

$$E(\hat{\beta}_{SRMLE}) = \beta, \tag{2.17}$$

and

$$\begin{aligned} Cov(\hat{\beta}_{SRMLE}) &= C^{-1} - C^{-1}H'(\Psi + HC^{-1}H')^{-1}HC^{-1} \\ &= (C + H'\Psi^{-1}H)^{-1}. \end{aligned} \tag{2.18}$$

To improve the performance of the SRMLE, Varathan and Wijekoon (2016) proposed the stochastic restricted ridge maximum likelihood estimator (SRRMLE) by incorporating the Ridge estimator to the logistic regression (LRE) by replacing MLE in (2.16), and have obtained the following estimator:

$$\hat{\beta}_{SRRMLE} = \hat{\beta}_{LRE} + C^{-1}H'(\Psi + HC^{-1}H')^{-1}(h - H\hat{\beta}_{LRE}) \tag{2.19}$$

where,

$$\begin{aligned} \hat{\beta}_{LRE} &= (X'\hat{W}X + kI)^{-1}X'\hat{W}X\hat{\beta}_{MLE} \\ &= (C + kI)^{-1}C\hat{\beta}_{MLE} \\ &= Z_k\hat{\beta}_{MLE} \end{aligned} \tag{2.20}$$

where $Z_k = (C + kI)^{-1}C$ and k is a constant, $k \geq 0$.

Further,

$$\begin{aligned} E[\hat{\beta}_{LRE}] &= Z_k\beta; \quad \text{and} \\ Cov[\hat{\beta}_{LRE}] &= Z_k(C + kI)^{-1}. \end{aligned} \tag{2.21}$$

The asymptotic properties of SRRMLE:

$$\begin{aligned} E(\hat{\beta}_{SRRMLE}) &= E[\hat{\beta}_{LRE}] + C^{-1}H'(\Psi + HC^{-1}H')^{-1}(H\beta - HE(\hat{\beta}_{LRE})) \\ &= Z_k\beta + C^{-1}H'(\Psi + HC^{-1}H')^{-1}(H\beta - HZ_k\beta) \\ &= [Z_k + C^{-1}H'(\Psi + HC^{-1}H')^{-1}(H - HZ_k)]\beta \end{aligned} \tag{2.22}$$

$$
\begin{aligned}
Cov(\hat{\beta}_{SRRMLE}) &= Cov[\hat{\beta}_{LRE} + C^{-1}H'(\Psi + HC^{-1}H')^{-1}(h - H\hat{\beta}_{LRE})] \\
&= Z_k(C+kI)^{-1} + C^{-1}H'(\Psi + HC^{-1}H')^{-1} \\
&\quad [\Psi + HZ_k(C+kI)^{-1}H'](\Psi + HC^{-1}H')^{-1}HC^{-1} \\
&\quad -2C^{-1}H'(\Psi + HC^{-1}H')^{-1}HZ_k(C+kI)^{-1} \qquad (2.23)
\end{aligned}
$$

Moreover, it was concluded in their paper that the estimator SRRMLE is superior over MLE, LRE and SRMLE under certain conditions.

3 The new proposed estimator

By incorporating the Liu estimator to the logistic regression under linear stochastic restriction, we propose a new biased estimator which is called stochastic restricted liu maximum likelihood estimator (SRLMLE) as follows:

$$\hat{\beta}_{SRLMLE} = Z_d\hat{\beta}_{SRMLE}. \qquad (3.1)$$

Note that the asymptotic properties of SRLMLE are given as

$$
\begin{aligned}
E(\hat{\beta}_{SRLMLE}) &= Z_d E[\hat{\beta}_{SRMLE}] \\
&= Z_d\beta, \qquad (3.2)
\end{aligned}
$$

and

$$
\begin{aligned}
Cov(\hat{\beta}_{SRLMLE}) &= Cov[Z_d\hat{\beta}_{SRMLE}] \\
&= Z_d Var[\hat{\beta}_{SRMLE}]Z_d' \\
&= Z_d(C + H'\Psi^{-1}H)^{-1}Z_d' \\
&= (C+I)^{-1}(C+dI)(C+H'\Psi^{-1}H)^{-1}(C+dI)(C+I)^{-1} \\
&= A_d \ (say). \qquad (3.3)
\end{aligned}
$$

respectively. Now we consider the performance of this new estimator over the existing estimators given in the literature.

4 Mean square error matrix criteria

The mean square error (MSE) of estimator $\hat{\beta}$ which is an estimator of β is defined as follows:

$$
\begin{aligned}
MSE(\hat{\beta}, \beta) &= E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] \\
&= D(\hat{\beta}) + B(\hat{\beta})B'(\hat{\beta}) \qquad (4.1)
\end{aligned}
$$

where $D(\hat{\beta})$ is the dispersion matrix, and $B(\hat{\beta}) = E(\hat{\beta}) - \beta$ denotes the bias vector.

The scalar mean square error (SMSE) of the estimator $\hat{\beta}$ can be defined as

$$SMSE(\hat{\beta}, \beta) = trace[MSE(\hat{\beta}, \beta)] \qquad (4.2)$$

For two given estimators $\hat{\beta}_1$ and $\hat{\beta}_2$, the estimator $\hat{\beta}_2$ is said to be superior to $\hat{\beta}_1$ under the MSE criterion if and only if

$$M(\hat{\beta}_1, \hat{\beta}_2) = MSE(\hat{\beta}_1, \beta) - MSE(\hat{\beta}_2, \beta) \geq 0. \tag{4.3}$$

The mean square error of the proposed estimator SRLMLE is

$$\begin{aligned} MSE(\hat{\beta}_{SRLMLE}) &= D(\hat{\beta}_{SRLMLE}) + B(\hat{\beta}_{SRLMLE})B'(\hat{\beta}_{SRLMLE}) \\ &= A_d + \delta\delta' \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \delta &= Bias(\hat{\beta}_{SRLMLE}) \\ &= E(\hat{\beta}_{SRLMLE}) - \beta \\ &= (Z_d - I)\beta. \end{aligned} \tag{4.5}$$

Hence, the scalar mean square error of $\hat{\beta}_{SRLMLE}$ is

$$SMSE(\hat{\beta}_{SRLMLE}) = trace\{A_d + \delta\delta'\} \tag{4.6}$$

5 The performance of the new estimator

In this section we investigate the theoretical performance of the proposed estimator SRLMLE over some existing estimators: MLE, LLE, RLLE, and SRMLE with respect to the mean square error sense. Further, the necessary conditions for superiority of the proposed estimator compared with the other existing estimators are also derived.

- **SRLMLE Versus MLE**

$$\begin{aligned} MSE(\hat{\beta}_{MLE}) - MSE(\hat{\beta}_{SRLMLE}) &= \{D(\hat{\beta}_{MLE}) - D(\hat{\beta}_{SRLMLE})\} \\ &\quad + \{B(\hat{\beta}_{MLE})B'(\hat{\beta}_{MLE}) - B(\hat{\beta}_{SRLMLE})B'(\hat{\beta}_{SRLMLE})\} \\ &= C^{-1} - \{A_d + \delta\delta'\} \\ &= M_1 - N_1 \end{aligned} \tag{5.1}$$

where $M_1 = C^{-1}$ and $N_1 = A_d + \delta\delta'$. One can obviously say that A_d and M_1 are positive definite and $\delta\delta'$ is non-negative definite matrix. Further by Lemma 1 (see Appendix 1), it is clear that N_1 is positive definite matrix. By Lemma 2 (see Appendix 1), if $\lambda_{\max}(N_1 M_1^{-1}) < 1$, then $M_1 - N_1$ is a positive definite matrix, where $\lambda_{\max}(N_1 M_1^{-1})$ is the largest eigen value of $N_1 M_1^{-1}$. Based on the above arguments, the following theorem can be stated.

Theorem 1 *The estimator SRLMLE is superior to MLE if and only if* $\lambda_{\max}(N_1 M_1^{-1}) < 1$

- **SRLMLE Versus LLE**

$$\begin{aligned} MSE(\hat{\beta}_{LLE}) - MSE(\hat{\beta}_{SRLMLE}) &= \{D(\hat{\beta}_{LLE}) - D(\hat{\beta}_{SRLMLE})\} \\ &\quad + \{B(\hat{\beta}_{LLE})B'(\hat{\beta}_{LLE}) \\ &\quad - B(\hat{\beta}_{SRLMLE})B'(\hat{\beta}_{SRLMLE})\} \end{aligned} \tag{5.2}$$

Since $B(\hat{\beta}_{LLE}) = B(\hat{\beta}_{SRLMLE}) = (Z_d - I)\beta$, then the Eq. (5.2) becomes

$$\begin{aligned} MSE(\hat{\beta}_{LLE}) - MSE(\hat{\beta}_{SRLMLE}) &= \{D(\hat{\beta}_{LLE}) - D(\hat{\beta}_{SRLMLE})\} \\ &= Z_d C^{-1} Z_d' - Z_d (C + H'\Psi^{-1}H)^{-1} Z_d' \\ &= Z_d \{C^{-1} - (C + H'\Psi^{-1}H)^{-1}\} Z_d' \\ &= Z_d \{C^{-1} H' (\Psi + HC^{-1}H')^{-1} HC^{-1}\} Z_d' \\ &= Z_d M_2 Z_d' \end{aligned} \tag{5.3}$$

where $Z_d = (C + I)^{-1}(C + dI)$ and $M_2 = \{C^{-1}H'(\Psi + HC^{-1}H')^{-1}HC^{-1}\}$. One can obviously say that Z_d and M_2 are positive definite matrices. Consequently, $Z_d M_2 Z_d'$ is a positive definite matrix. Therefore, the estimator $SRLMLE$ is always superior to LLE in the sense of MSE.

- **SRLMLE Versus SRMLE**

$$\begin{aligned} MSE(\hat{\beta}_{SRMLE}) - MSE(\hat{\beta}_{SRLMLE}) &= \{D(\hat{\beta}_{SRMLE}) - D(\hat{\beta}_{SRLMLE})\} \\ &\quad + \{B(\hat{\beta}_{SRMLE})B'(\hat{\beta}_{SRMLE}) - B(\hat{\beta}_{SRLMLE})B'(\hat{\beta}_{SRLMLE})\} \\ &= (C + H'\Psi^{-1}H)^{-1} - \{A_d + \delta\delta'\} \\ &= M_3 - N_3 \end{aligned} \tag{5.4}$$

where $M_3 = (C + H'\Psi^{-1}H)^{-1}$ and $N_3 = A_d + \delta\delta'$. One can obviously say that A_d and M_3 are positive definite and $\delta\delta'$ is non-negative definite matrices. Further by Lemma 1, it is clear that N_3 is positive definite matrix. By Lemma 2 (see Appendix 1), if $\lambda_{\max}(N_3 M_3^{-1}) < 1$, then $M_3 - N_3$ is a positive definite matrix, where $\lambda_{\max}(N_3 M_3^{-1})$ is the the largest eigen value of $N_3 M_3^{-1}$. Based on the above arguments, the following theorem can be stated.

Theorem 2 *The estimator $SRLMLE$ is superior to $SRMLE$ if and only if $\lambda_{\max}(N_3 M_3^{-1}) < 1$.*

- **SRLMLE Versus RLLE**

$$\begin{aligned} MSE(\hat{\beta}_{RLLE}) - MSE(\hat{\beta}_{SRLMLE}) &= \{D(\hat{\beta}_{RLLE}) - D(\hat{\beta}_{SRLMLE})\} \\ &\quad + \{B(\hat{\beta}_{RLLE})B'(\hat{\beta}_{RLLE}) - B(\hat{\beta}_{SRLMLE})B'(\hat{\beta}_{SRLMLE})\} \\ &= \{Z_d A Z_d' - A_d\} + \{\delta_3\delta_3' - \delta\delta'\} \end{aligned} \tag{5.5}$$

Now consider,

$$\begin{aligned} D(\hat{\beta}_{RLLE}) - D(\hat{\beta}_{SRLMLE}) &= Z_d A Z_d' - A_d \\ &= M_4 - N_4 \\ &= D_2^* \ (say) \end{aligned} \tag{5.6}$$

where $M_4 = Z_d A Z_d'$ and $N_4 = A_d$. One can obviously say that M_4 and N_4 are positive definite matrices. By Lemma 2, if $\lambda_{\max}(N_4 M_4^{-1}) < 1$, then $D_2^* = M_4 - N_4$

is a positive definite matrix, where $\lambda_{\max}(N_4M_4^{-1})$ is the the largest eigen value of $N_4M_4^{-1}$. Based on the above arguments and Lemma 3, the following theorem can be stated.

Theorem 3 *When $\lambda_{\max}(N_4M_4^{-1}) < 1$, the estimator SRLMLE is superior to RLLE if and only if $\delta'(D_2^* + \delta_3'\delta_3)^{-1}\delta \leq 1$.*

Based on the above results one can say that the new estimator SRLMLE is always superior to LLE in the MSE sense. Moreover, the SRLMLE is superior over MLE, SRMLE and RLLE with respect to the mean squared error matrix sense under certain conditions. The comparison of the performances of these estimators is done by a simulation study in the following section.

6 A simulation study

A Monte Carlo simulation is conducted to illustrate the performance of the estimators MLE, LLE, SRMLE, RLLE and SRLMLE by means of Scalar Mean Square Error (SMSE). Following McDonald and Galarneau (1975) and Kibria et al. (2003), the explanatory variables are generated using the following equation.

$$x_{ij} = (1-\rho^2)^{1/2} z_{ij} + \rho z_{i,p+1}, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, p \tag{6.1}$$

where z_{ij} are pseudo- random numbers from standardized normal distribution and ρ^2 represents the correlation between any two explanatory variables. Four explanatory variables are generated using (6.1). Four different values of ρ corresponding to 0.70, 0.80, 0.90 and 0.99 are considered. Further for the sample size n, four different values 25, 50, 75, and 100 are also considered. The dependent variable y_i in (2.1) is obtained from the Bernoulli(π_i) distribution where $\pi_i = \frac{\exp(x_i'\beta)}{1+\exp(x_i'\beta)}$. The parameter values of $\beta_1, \beta_2, \ldots, \beta_p$ are chosen so that $\sum_{j=1}^{p} \beta_j^2 = 1$ and $\beta_1 = \beta_2 = \cdots = \beta_p$.

Moreover, we choose the following restrictions.

$$H = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.2}$$

Following Wu and Asar (2015) and Mansson et al. (2012), we can theoretically obtain the optimum value of the biasing parameter d by minimizing SMSE values with respect to d. However, for simplicity in this paper we select some values of d in the range $0 < d < 1$.

The simulation is repeated 3000 times by generating new pseudo- random numbers and the simulated SMSE values of the estimators are obtained using the following equation.

$$S\hat{M}SE(\hat{\beta}^*) = \frac{1}{3000} \sum_{r=1}^{3000} (\hat{\beta}_r - \beta)'(\hat{\beta}_r - \beta) \tag{6.3}$$

where $\hat{\beta}_r$ is any estimator considered in the rth simulation. The results of the simulation are reported in Tables 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 ,17, and 18 (Appendix 3) and also displayed in Figs. 1, 2, 3, and 4 (Appendix 2). According to Figs. 1, 2, 3, and 4, it can be observed that in general increase in degree of correlation between two explanatory variables ρ inflates the estimated SMSE of all the estimators and increase in sample size n declines the estimated SMSE of all the estimators. Further it was noted from the Tables 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 , 17, and 18 that when ρ = 0.7, 0.8, and 0.9 the new estimator SRLMLE performed well compared to all the other estimators MLE, LLE, RLLE and SRMLE with respect to all values of d in the range $0 < d < 1$ and n= 25, 50, 75 and 100. Moreover, in the case of $\rho = 0.99$ and n = 25, 50, the estimator RLLE produces smaller SMSE values compared to all the other estimators considered in this study.

7 Numerical example

In this section we apply the new estimator SRLMLE to the data set taken from the official web site (http://www.scb.se/) of the Statistics Sweden to illustrate the theoretical results. The same data set was used by Wu and Asar (2016), Asar and Genc (2015) and Mansson et al. (2012) as a numerical example. The data set has 271 observations corresponding to the municipalities of Sweden, and the available variables are:

$x1$: Population,
$x2$: Number of unemployed people,
$x3$: Number of newly constructed buildings,
$x4$: Number of bankrupt firms,
y: Net population change which is defined as

$$y = \begin{cases} 1; & \text{if there is an increase in the population;} \\ 0; & \text{Otherwise.} \end{cases}$$

The correlation matrix of the predictor variables is given in Table 1. Since the bivariate correlations (Table 1) among all predictor variables are very high (greater than 0.9), and the condition number is 38.3274 indicates a severe multicollinearity problem in this data set. The SMSE values of MLE, LLE, RLLE, SRMLE, and SRLMLE for a selected values of biasing parameter d in the range $0 < d < 1$ is given in the Table 2. Note that the SMSE values of the new estimator SRLMLE are minimum when comparing the same with the other estimators for all values of d in the range $0 < d < 1$. Therefore the SRLMLE performs well with compared to the estimators

Table 1 The correlation matrix of the design matrix

	$x1$	$x2$	$x3$	$x4$
$x1$	1.0000	0.9939	0.9481	0.9324
$x2$	0.9939	1.0000	0.9288	0.9016
$x3$	0.9481	0.9288	1.0000	0.9463
$x4$	0.9324	0.9016	0.9463	1.0000

Table 2 The SMSE values of estimators for different values of d

	MLE	LLE	RLLE	SRMLE	SRLMLE
$d = 0.05$	0.000411363	0.000411062	2.281557167	0.000411165	**0.000410865**
$d = 0.1$	0.000411363	0.000411078	2.281562599	0.000411165	**0.000410881**
$d = 0.2$	0.000411363	0.000411110	2.281573462	0.000411165	**0.000410912**
$d = 0.3$	0.000411363	0.000411142	2.281584326	0.000411165	**0.000410944**
$d = 0.4$	0.000411363	0.000411174	2.281595190	0.000411165	**0.000410975**
$d = 0.5$	0.000411363	0.000411206	2.281606054	0.000411165	**0.000411007**
$d = 0.6$	0.000411363	0.000411238	2.281616919	0.000411165	**0.000411039**
$d = 0.7$	0.000411363	0.000411269	2.281627784	0.000411165	**0.000411070**
$d = 0.8$	0.000411363	0.000411301	2.281638650	0.000411165	**0.000411102**
$d = 0.9$	0.000411363	0.000411333	2.281649516	0.000411165	**0.000411134**
$d = 0.99$	0.000411363	0.000411362	2.281659296	0.000411165	**0.000411162**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

MLE, LLE, RLLE SRMLE in the SMSE sense. Further, the SMSE of the estimator RLLE is the largest among all estimators in the given range of d. The findings of this Numerical application coincides with the simulation results displayed in Table 17, which is the case of $n = 100$ and $\rho = 0.90$.

8 Concluding remarks

In this paper, we proposed the Stochastic Restricted Liu Maximum Likelihood Estimator (SRLMLE) for logistic regression model when the linear stochastic restriction is available. The relative performance of the proposed estimator SRLMLE over MLE, LLE, RLLE and SRMLE were analyzed by a numerical example and a Monte Carlo simulation study. The empirical results of this paper show that the proposed estimator SRLMLE is always superior to the LLE. Also, it is superior over MLE, RLLE and SRMLE with respect to all the values of n, ρ, and $0 < d < 1$ except the case of very high degree of collinearity $\rho = 0.99$ and the low sample size $n = 25, 50$.

Appendix 1

Lemma 1 *Let $A : n \times n$ and $B : n \times n$ such that $A > 0$ and $B \geq 0$. Then $A + B > 0$* (Rao and Toutenburg 1995).

Lemma 2 *Let the two $n \times n$ matrices $M > 0$,$N \geq 0$, then $M > N$ if and only if $\lambda_{\max}(NM^{-1}) < 1$* (Rao et al. 2008).

Lemma 3 *Let $\tilde{\beta}_j = A_j y$, $j = 1, 2$ be two competing homogeneous linear estimators of β. Suppose that $D = Cov(\tilde{\beta}_1) - Cov(\tilde{\beta}_2) > 0$, where $Cov(\tilde{\beta}_j)$, $j = 1, 2$ denotes the covariance matrix of $\tilde{\beta}_j$. Then $\Delta(\tilde{\beta}_1, \tilde{\beta}_2) = MSEM(\tilde{\beta}_1) - MSEM(\tilde{\beta}_2) \geq 0$ if and only if $d_2'(D + d_1' d_1)^{-1} d_2 \leq 1$, where $MSEM(\tilde{\beta}_j)$, d_j; $j = 1, 2$ denote the Mean*

Square Error Matrix and bias vector of $\tilde{\beta}_j$, *respectively* (Trenkler and Toutenburg 1990).

Appendix 2

See Figs. 1, 2, 3, and 4.

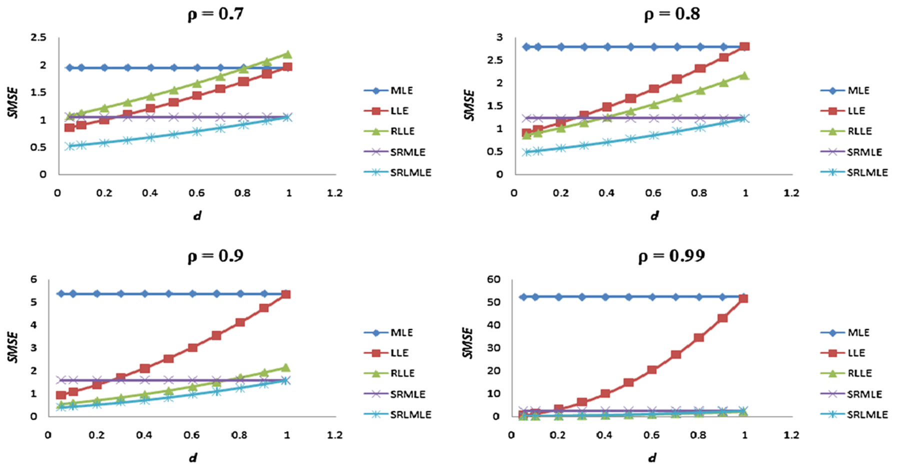

Fig. 1 Estimated SMSE values for MLE, LLE, RLLE, SRMLE and SRLMLE for n = 25

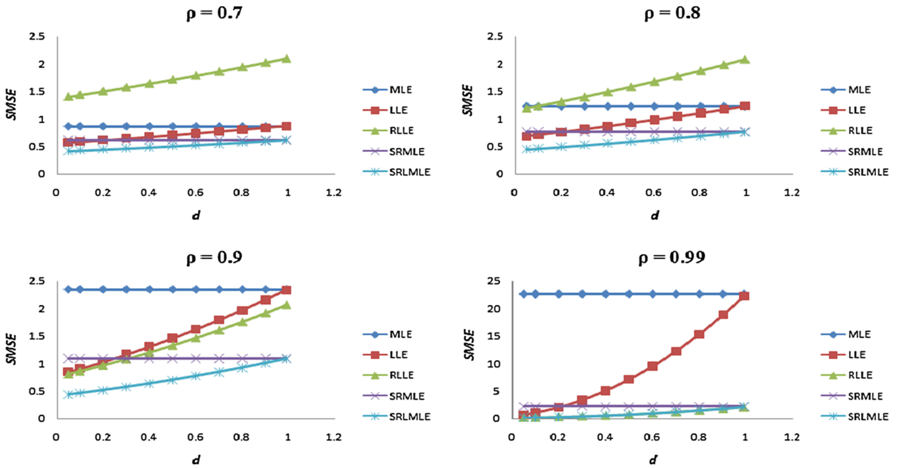

Fig. 2 Estimated SMSE values for MLE, LLE, RLLE, SRMLE and SRLMLE for n = 50

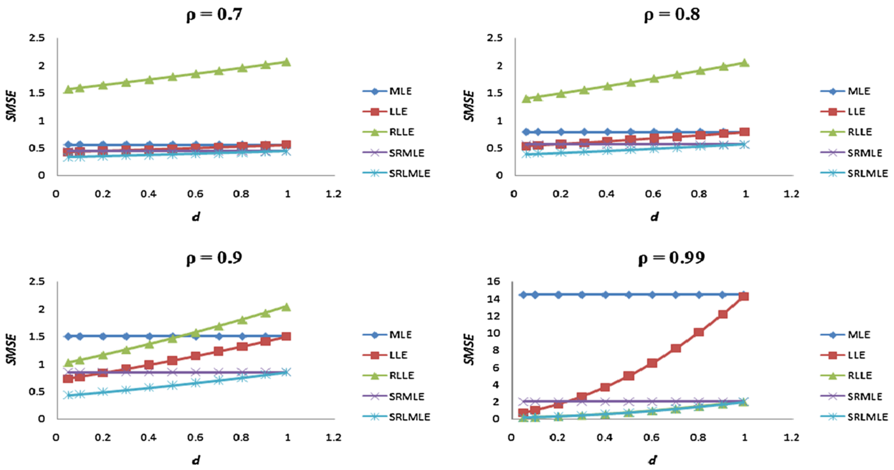

Fig. 3 Estimated SMSE values for MLE, LLE, RLLE, SRMLE and SRLMLE for n = 75

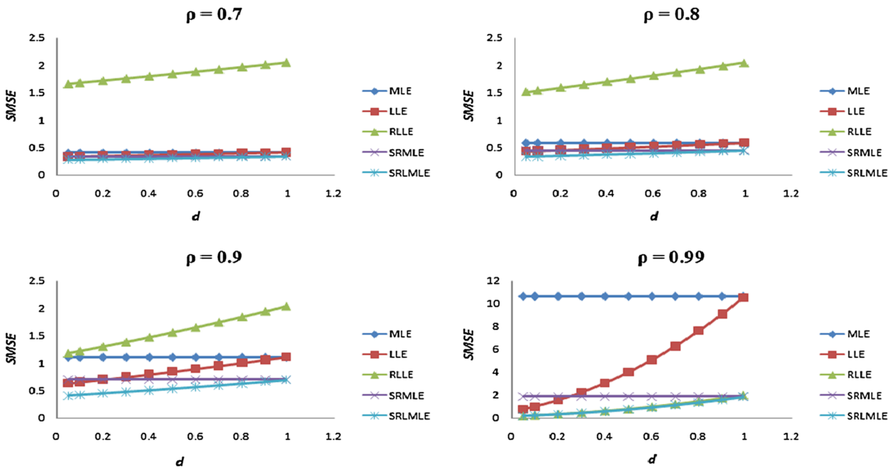

Fig. 4 Estimated SMSE values for MLE, LLE, RLLE, SRMLE and SRLMLE for n = 100

Appendix 3

See Tables 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 ,17, and 18.

Table 3 The estimated MSE values for different d when $n = 25$ and $\rho = 0.70$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	1.9464	1.9464	1.9464	1.9464	1.9464	1.9464	1.9464	1.9464	1.9464	1.9464	1.9464
LLE	0.8555	0.9006	0.9954	1.0964	1.2036	1.3171	1.4368	1.5627	1.6948	1.8332	1.9630
RLLE	1.0765	1.1223	1.2184	1.3209	1.4296	1.5445	1.6657	1.7932	1.9269	2.0669	2.1982
SRMLE	1.0449	1.0449	1.0449	1.0449	1.0449	1.0449	1.0449	1.0449	1.0449	1.0449	1.0449
SRLMLE	**0.5225**	**0.5424**	**0.5847**	**0.6304**	**0.6794**	**0.7319**	**0.7877**	**0.8470**	**0.9096**	**0.9756**	**1.0378**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 4 The estimated MSE values for different d when $n = 25$ and $\rho = 0.80$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	2.7913	2.7913	2.7913	2.7913	2.7913	2.7913	2.7913	2.7913	2.7913	2.7913	2.7913
LLE	0.9221	0.9914	1.1401	1.3025	1.4785	1.6680	1.8712	2.0880	2.3184	2.5624	2.7936
RLLE	0.8581	0.9081	1.0149	1.1307	1.2555	1.3895	1.5324	1.6845	1.8456	2.0157	2.1766
SRMLE	1.2325	1.2325	1.2325	1.2325	1.2325	1.2325	1.2325	1.2325	1.2325	1.2325	1.2325
SRLMLE	**0.4958**	**0.5222**	**0.5792**	**0.6416**	**0.7095**	**0.7830**	**0.8619**	**0.9463**	**1.0362**	**1.1316**	**1.2222**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 5 The estimated MSE values for different d when $n = 25$ and $\rho = 0.90$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	5.3804	5.3804	5.3804	5.3804	5.3804	5.3804	5.3804	5.3804	5.3804	5.3804	5.3804
LLE	0.9533	1.0845	1.3813	1.7240	2.1126	2.5469	3.0272	3.5532	4.1251	4.7429	5.3381
RLLE	0.5393	0.5905	0.7045	0.8341	0.9794	1.1404	1.3170	1.5092	1.7171	1.9407	2.1552
SRMLE	1.5915	1.5915	1.5915	1.5915	1.5915	1.5915	1.5915	1.5915	1.5915	1.5915	1.5915
SRLMLE	**0.4063**	**0.4422**	**0.5227**	**0.6150**	**0.7192**	**0.8351**	**0.9628**	**1.1023**	**1.2535**	**1.4166**	**1.5734**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 6 The estimated MSE values for different d when $n = 25$ and $\rho = 0.99$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	52.3691	52.3691	52.3691	52.3691	52.3691	52.3691	52.3691	52.3691	52.3691	52.3691	52.3691
LLE	0.7093	1.3238	3.2545	6.1211	9.9235	14.6618	20.3359	26.9459	34.4917	42.9733	51.4069
RLLE	0.1709	0.1963	0.2736	0.3864	0.5346	0.7183	0.9374	1.1920	1.4819	1.8074	2.1306
SRMLE	2.4583	2.4583	2.4583	2.4583	2.4583	2.4583	2.4583	2.4583	2.4583	2.4583	2.4583
SRLMLE	0.1748	0.2039	0.2924	0.4213	0.5908	0.8008	1.0513	1.3423	1.6738	2.0458	2.4152

Table 7 The estimated MSE values for different d when $n = 50$ and $\rho = 0.70$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	0.8628	0.8628	0.8628	0.8628	0.8628	0.8628	0.8628	0.8628	0.8628	0.8628	0.8628
LLE	0.5724	0.5864	0.6148	0.6441	0.6742	0.7050	0.7367	0.7691	0.8024	0.8364	0.8677
RLLE	1.4065	1.4390	1.5053	1.5736	1.6437	1.7159	1.7899	1.8659	1.9437	2.0236	2.0970
SRMLE	0.6129	0.6129	0.6129	0.6129	0.6129	0.6129	0.6129	0.6129	0.6129	0.6129	0.6129
SRLMLE	**0.4178**	**0.4265**	**0.4445**	**0.4631**	**0.4825**	**0.5025**	**0.5232**	**0.5446**	**0.5667**	**0.5894**	**0.6105**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 8 The estimated MSE values for different d when $n = 50$ and $\rho = 0.80$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	1.2291	1.2291	1.2291	1.2291	1.2291	1.2291	1.2291	1.2291	1.2291	1.2291	1.2291
LLE	0.6844	0.7088	0.7593	0.8118	0.8664	0.9231	0.9817	1.0425	1.1053	1.1701	1.2302
RLLE	1.1988	1.2387	1.3210	1.4065	1.4952	1.5873	1.6826	1.7811	1.8830	1.9880	2.0854
SRMLE	0.7660	0.7660	0.7660	0.7660	0.7660	0.7660	0.7660	0.7660	0.7660	0.7660	0.7660
SRLMLE	**0.4466**	**0.4604**	**0.4891**	**0.5191**	**0.5504**	**0.5831**	**0.6170**	**0.6523**	**0.6889**	**0.7268**	**0.7620**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 9 The estimated MSE values for different d when $n = 50$ and $\rho = 0.90$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	2.3496	2.3496	2.3496	2.3496	2.3496	2.3496	2.3496	2.3496	2.3496	2.3496	2.3496
LLE	0.8470	0.9056	1.0296	1.1630	1.3057	1.4577	1.6190	1.7895	1.9694	2.1586	2.3367
RLLE	0.8157	0.8660	0.9721	1.0856	1.2066	1.3350	1.4710	1.6143	1.7651	1.9234	2.0722
SRMLE	1.0940	1.0940	1.0940	1.0940	1.0940	1.0940	1.0940	1.0940	1.0940	1.0940	1.0940
SRLMLE	**0.4358**	**0.4613**	**0.5154**	**0.5735**	**0.6357**	**0.7019**	**0.7722**	**0.8466**	**0.9250**	**1.0074**	**1.0851**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 10 The estimated MSE values for different d when $n = 50$ and $\rho = 0.99$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	22.6486	22.6486	22.6486	22.6486	22.6486	22.6486	22.6486	22.6486	22.6486	22.6486	22.6486
LLE	0.6811	1.0427	2.0309	3.3725	5.0672	7.1153	9.5167	12.2713	15.3793	18.8405	22.2576
RLLE	0.1377	0.1710	0.2608	0.3813	0.5324	0.7142	0.9267	1.1699	1.4438	1.7483	2.0486
SRMLE	2.1961	2.1961	2.1961	2.1961	2.1961	2.1961	2.1961	2.1961	2.1961	2.1961	2.1961
SRLMLE	0.1407	0.1759	0.2705	0.3976	0.5572	0.7492	0.9737	1.2306	1.5200	1.8418	2.1592

Table 11 The estimated MSE values for different d when $n = 75$ and $\rho = 0.70$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	0.5536	0.5536	0.5536	0.5536	0.5536	0.5536	0.5536	0.5536	0.5536	0.5536	0.5536
LLE	0.4225	0.4291	0.4424	0.4560	0.4698	0.4838	0.4980	0.5125	0.5273	0.5422	0.5559
RLLE	1.5723	1.5965	1.6456	1.6955	1.7464	1.7981	1.8505	1.9043	1.9588	2.0141	2.0647
SRMLE	0.4366	0.4366	0.4366	0.4366	0.4366	0.4366	0.4366	0.4366	0.4366	0.4366	0.4366
SRLMLE	**0.3366**	**0.3413**	**0.3509**	**0.3608**	**0.3708**	**0.3812**	**0.3918**	**0.4026**	**0.4137**	**0.4250**	**0.4354**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 12 The estimated MSE values for different d when $n = 75$ and $\rho = 0.80$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	0.7871	0.7871	0.7871	0.7871	0.7871	0.7871	0.7871	0.7871	0.7871	0.7871	0.7871
LLE	0.5306	0.5428	0.5678	0.5935	0.6197	0.6466	0.6742	0.7023	0.7311	0.7606	0.7876
RLLE	1.3977	1.4291	1.4932	1.5589	1.6262	1.6952	1.7658	1.8380	1.9119	1.9873	2.0567
SRMLE	0.5618	0.5618	0.5618	0.5618	0.5618	0.5618	0.5618	0.5618	0.5618	0.5618	0.5618
SRLMLE	**0.3858**	**0.3939**	**0.4105**	**0.4277**	**0.4454**	**0.4635**	**0.4822**	**0.5013**	**0.5210**	**0.5411**	**0.5597**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 13 The estimated MSE values for different d when $n = 75$ and $\rho = 0.90$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	1.5018	1.5018	1.5018	1.5018	1.5018	1.5018	1.5018	1.5018	1.5018	1.5018	1.5018
LLE	0.7294	0.7627	0.8318	0.9042	0.9800	1.0591	1.1416	1.2275	1.3167	1.4093	1.4955
RLLE	1.0321	1.0766	1.1687	1.2650	1.3657	1.4706	1.5798	1.6933	1.8111	1.9331	2.0466
SRMLE	0.8478	0.8478	0.8478	0.8478	0.8478	0.8478	0.8478	0.8478	0.8478	0.8478	0.8478
SRLMLE	**0.4300**	**0.4478**	**0.4849**	**0.5238**	**0.5646**	**0.6072**	**0.6516**	**0.6979**	**0.7460**	**0.7960**	**0.8425**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 14 The estimated MSE values for different d when $n = 75$ and $\rho = 0.99$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	14.4586	14.4586	14.4586	14.4586	14.4586	14.4586	14.4586	14.4586	14.4586	14.4586	14.4586
LLE	0.7197	0.9961	1.6978	2.5981	3.6971	4.9946	6.4908	8.1856	10.0790	12.1710	14.2236
RLLE	0.1541	0.1935	0.2926	0.4187	0.5717	0.7517	0.9586	1.1924	1.4532	1.7409	2.0229
SRMLE	2.0210	2.0210	2.0210	2.0210	2.0210	2.0210	2.0210	2.0210	2.0210	2.0210	2.0210
SRLMLE	**0.1506**	**0.1892**	**0.2862**	**0.4099**	**0.5602**	**0.7371**	**0.9407**	**1.1708**	**1.4276**	**1.7110**	**1.9888**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 15 The estimated MSE values for different d when $n = 100$ and $\rho = 0.70$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	0.4091	0.4091	0.4091	0.4091	0.4091	0.4091	0.4091	0.4091	0.4091	0.4091	0.4091
LLE	0.3342	0.3381	0.3458	0.3536	0.3616	0.3696	0.3777	0.3859	0.3943	0.4027	0.4103
RLLE	1.6640	1.6833	1.7223	1.7617	1.8017	1.8423	1.8833	1.9249	1.9670	2.0096	2.0484
SRMLE	0.3405	0.3405	0.3405	0.3405	0.3405	0.3405	0.3405	0.3405	0.3405	0.3405	0.3405
SRLMLE	**0.2797**	**0.2826**	**0.2885**	**0.2946**	**0.3008**	**0.3071**	**0.3135**	**0.3201**	**0.3268**	**0.3336**	**0.3398**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 16 The estimated MSE values for different d when $n = 100$ and $\rho = 0.80$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	0.5813	0.5813	0.5813	0.5813	0.5813	0.5813	0.5813	0.5813	0.5813	0.5813	0.5813
LLE	0.4314	0.4388	0.4538	0.4690	0.4845	0.5003	0.5163	0.5326	0.5493	0.5661	0.5816
RLLE	1.5160	1.5418	1.5941	1.6475	1.7018	1.7571	1.8133	1.8706	1.9288	1.9880	2.0422
SRMLE	0.4461	0.4461	0.4461	0.4461	0.4461	0.4461	0.4461	0.4461	0.4461	0.4461	0.4461
SRLMLE	**0.3344**	**0.3397**	**0.3505**	**0.3616**	**0.3730**	**0.3845**	**0.3964**	**0.4084**	**0.4208**	**0.4333**	**0.4448**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 17 The estimated MSE values for different d when $n = 100$ and $\rho = 0.90$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	1.1084	1.1084	1.1084	1.1084	1.1084	1.1084	1.1084	1.1084	1.1084	1.1084	1.1084
LLE	0.6327	0.6543	0.6986	0.7444	0.7919	0.8410	0.8917	0.9439	0.9978	1.0533	1.1045
RLLE	1.1828	1.2218	1.3021	1.3851	1.4709	1.5595	1.6509	1.7451	1.8421	1.9419	2.0341
SRMLE	0.6980	0.6980	0.6980	0.6980	0.6980	0.6980	0.6980	0.6980	0.6980	0.6980	0.6980
SRLMLE	**0.4080**	**0.4210**	**0.4478**	**0.4756**	**0.5044**	**0.5342**	**0.5649**	**0.5967**	**0.6295**	**0.6632**	**0.6944**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

Table 18 The estimated MSE values for different d when $n = 100$ and $\rho = 0.99$

	d = 0.05	d = 0.1	d = 0.2	d = 0.3	d = 0.4	d = 0.5	d = 0.6	d = 0.7	d = 0.8	d = 0.9	d = 0.99
MLE	10.6602	10.6602	10.6602	10.6602	10.6602	10.6602	10.6602	10.6602	10.6602	10.6602	10.6602
LLE	0.7582	0.9867	1.5411	2.2257	3.0405	3.9854	5.0604	6.2656	7.6009	9.0664	10.4966
RLLE	0.1849	0.2287	0.3344	0.4639	0.6175	0.7949	0.9964	1.2217	1.4711	1.7443	2.0107
SRMLE	1.8841	1.8841	1.8841	1.8841	1.8841	1.8841	1.8841	1.8841	1.8841	1.8841	1.8841
SRLMLE	**0.1710**	**0.2112**	**0.3082**	**0.4274**	**0.5689**	**0.7325**	**0.9184**	**1.1265**	**1.3569**	**1.6094**	**1.8557**

The bold values indicate the exact places, where the proposed estimator is superior over the other estimators

References

Aguilera AM, Escabias M, Valderrama MJ (2006) Using principal components for estimating logistic regression with high-dimensional multicollinear data. Comput Stat Data Anal 50:1905–1924

Asar Y, Genc A (2015) New shrinkage parameters for the Liu-type logistic estimators. Commun Stat Simul Comput. doi:10.1080/03610918.2014.995815

Duffy DE, Santner TJ (1989) On the small sample prosperities of norm-restricted maximum likelihood estimators for logistic regression models. Commun Stat Theor Meth 18:959–980

Inan D, Erdogan BE (2013) Liu-type logistic estimator. Commun Stat Simul Comput 42:1578–1586

Kibria BMG (2003) Performance of some new ridge regression estimators. Commun Stat Theor Meth 32:419–435

Liu K (1993) A new class of biased estimate in linear regression. Commun Stat Theor Meth 22:393–402

Mansson G, Kibria BMG, Shukur G (2012) On Liu estimators for the logit regression model. The Royal Institute of Techonology, Centre of Excellence for Science and Innovation Studies (CESIS), Sweden, Paper No. 259

McDonald GC, Galarneau DI (1975) A Monte Carlo evaluation of some ridge type estimators. J Am Stat Assoc 70:407–416

Nja ME, Ogoke UP, Nduka EC (2013) The logistic regression model with a modified weight function. J Stat Econom Method 2(4):161–171

Rao CR, Toutenburg H, Shalabh, Heumann C (2008) Linear models and generalizations. Springer, Berlin

Rao CR, Toutenburg H (1995) Linear models: least squares and alternatives, 2nd edn. Springer, New York

Schaefer RL, Roi LD, Wolfe RA (1984) A ridge logistic estimator. Commun Stat Theor Meth 13:99–113

Şiray GU, Toker S, Kaçiranlar S (2015) On the restricted Liu estimator in logistic regression model. Commun Stat Simul Comput 44:217–232

Theil H, Goldberger AS (1961) On pure and mixed estimation in economics. Int Econ Rev 2(1):65–77. doi:10.2307/2525589

Trenkler G, Toutenburg H (1990) Mean square error matrix comparisons between biased estimators: an overview of recent results. Stat Pap 31:165–179. doi:10.1007/BF02924687

Urgan NN, Tez M (2008) Liu estimator in logistic regression when the data are collinear. In: International conference "continuous optimization and knowledge-based technologies", pp 323–327

Nagarajah V, Wijekoon P (2015) Stochastic restricted maximum likelihood estimator in logistic regression model. Open J Stat 5:837–851. doi:10.4236/ojs.2015.57082

Varathan N, Wijekoon P (2016) Ridge estimator in logistic regression under stochastic linear restriction. Br J Math Comput Sci 15(3):1. doi:10.9734/BJMCS/2016/24585

Wu J (2015) Modified restricted Liu estimator in logistic regression model. Comput Stat. doi:10.1007/s00180-015-0609-3

Wu J, Asar Y (2016) On almost unbiased ridge logistic estimator for the logistic regression model. Hacet J Math Stat 45(3):989–998. doi:10.15672/HJMS.20156911030

Wu J, Asar Y (2015) More on the restricted Liu estimator in the logistic regression model. Commun Stat Simul Comput. doi:10.1080/03610918.2015.1100735